Propositional Logic

A **proposition** is a statement or a sentence that is either *true* or *false*, but not both. See the examples below; each one of them is a proposition:

- Earth is a planet.

  This proposition is always *true*.

- The sun and the moon are the same objects.

  This proposition is always *false*.

- All my students enjoy my course.

  The truth value of this proposition varies based on several conditions but it will have one of the two values *true* or *false*.

- I will get 'A' in this course.

  The reader is urged to find the truth value of this proposition.

Propositions can be combined using one or more of the logical operators listed below.

Symbol	**Name**	**Synopsis**
$\neg$	negation	$\neg\ p$
$\wedge$	conjunction	$p \wedge q$
$\vee$	disjunction	$p \vee q$
$\Rightarrow$	implication	$p \Rightarrow q$
		$\neg\ p \vee q$
$\Leftrightarrow$	equivalence	$p \Leftrightarrow q$
		$(p \Rightarrow q) \wedge (q \Rightarrow p)$

The synopsis in the above table uses variable names to indicate the whole proposition. For example, a variable p may indicate the proposition "My wallet was stolen".
The truth value of a proposition involving logical operators can be evaluated using a truth table. See the example below:

p	q	$\neg\ p$	$\neg\ q$	$p \wedge q$	$\neg\ (p \wedge q)$	$\neg\ p \vee \neg\ q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Two propositions are said to be **logically equivalent** if and only if they are evaluated to the same truth value. In the truth table shown above, the two propositions $\neg\ (p \wedge q)$ and $\neg\ p \vee \neg\ q$ evaluate to the same truth values (see the two columns in the truth table corresponding to these two propositions) and hence they are logically equivalent. Below is another example which shows that the propositions $p \wedge (p \vee q)$ and $p \vee (p \wedge q)$ are logically equivalent.

p	q	$p \wedge q$	$p \vee q$	$p \wedge (p \vee q)$	$p \vee (p \wedge q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	F	F	F

As an exercise, check the logical equivalence of the following propositions

1. $p \Rightarrow (q \Rightarrow r)$ and $(p \Rightarrow q) \Rightarrow r$
2. $p \Rightarrow (q \vee r)$, $(p \wedge \neg\, q) \Rightarrow r$ and $(p \wedge \neg\, r) \Rightarrow q$

A proposition is said to be a **tautology** if it is always evaluated to *true* regardless of the truth values of the variables in the proposition.
Example: Let p be a proposition. Then $p \vee \neg\, p$ is a tautology.
A proposition is said to be a **contradiction** if it is always evaluated to *false* regardless of the truth values of the variables in the proposition.
Example: Let p be a proposition. Then $p \wedge \neg\, p$ is a contradiction.

Given the propositions p, q and r, the following logical equivalences hold. The symbol $\equiv$ stands for logical equivalence.

Name of the law		
Commutative law	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative law	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee q \equiv p \vee (q \vee r)$
Distributive law	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity law	$p \wedge \mathit{true} \equiv p$	$p \vee \mathit{false} \equiv p$
Negation law	$p \wedge \neg\, p \equiv \mathit{false}$	$p \vee \neg\, p \equiv \mathit{true}$
Double Negation law	$\neg\, (\neg\, p) \equiv p$	
Idempotent law	$p \wedge p \equiv p$	$p \vee p \equiv p$
DeMorgan's law	$\neg\, (p \wedge q) \equiv \neg\, p \vee \neg\, q$	$\neg\, (p \vee q) \equiv \neg\, p \wedge \neg\, q$
Universal bound law	$p \wedge \mathit{false} \equiv \mathit{false}$	$p \vee \mathit{true} \equiv \mathit{true}$
Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$

Converse and Inverse Statements

The converse of a conditional statement (the one that uses implication) of the form $p \Rightarrow q$ is the statement $q \Rightarrow p$. It is important to notice that a statement and its converse are **not** logically equivalent. See the example given below.

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

The notion of converse is important because when a statement written in natural language (such as English) is translated into a proposition, the person who is translating may misunderstand the statement and create the converse of the original statement. This is referred to as **converse error**. For example, let
p stand for the statement "I drink coffee" and
q stand for the statement "I get headache".
In this case, $p \Rightarrow q$ means that "If I drink coffee, I will get headache". On the other hand, $q \Rightarrow p$ mean that "If I have headache, I drink coffee". The original statement has the intention that I should not drink coffee whereas the converse indicates that I need coffee in order to relieve from headache. Converse errors such as this will lead to misunderstanding of requirements in a software system and consequently may lead to the development of incorrect software.

As an exercise, consider the following two statements and check whether they are logically equivalent; you need to convert them into propositional symbols before you check their logical equivalence.

- If you paid the full price for the book, then you did not buy at Amazon.
- You did not buy the book at Amazon or you paid the full price for it.

The inverse of a conditional statement (the one that uses implication) of the form $p \Rightarrow q$ is the statement $\neg\, p \Rightarrow \neg\, q$. Similar to the converse, it is also important to notice that the original statement and its inverse are **not** logically equivalent. See the example below.

p	q	$\neg\, p$	$\neg\, q$	$p \Rightarrow q$	$\neg\, p \Rightarrow \neg\, q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Like converse error, an inverse error may also lead to misunderstanding of the requirements during software development, and may eventually lead to an incorrect software. For example, consider the statement "If I drink coffee, I will get headache" which is of the form $p \Rightarrow q$. The inverse of this statement $\neg\, p \Rightarrow \neg\, q$ is "If I do not drink coffee, I will not get headache". If the first statement is true, the inverse need not be true because I may get headache through any other means.
As an exercise, consider the statement "If n is divisible by 6, then n is divisible by 2 and n is divisible by 3." Rewrite in logical form, also write its inverse and show that the statement and its inverse are not logically equivalent.

Rules of Inference
Given a set of statements called *premises*, the following inference rules can be used to derive additional statements and their truth values.

Name of the rule	**Premises**	**Conclusion**
Modus ponens	$p \Rightarrow q$ p	q
Modus tollens	$p \Rightarrow q$ $\neg q$	$\neg p$
Disjunctive addition	p	$p \vee q$
	q	$p \vee q$
Conjunctive simplification	$p \wedge q$	p
	$p \wedge q$	q
Conjunctive addition	p q	$p \wedge q$
Disjunctive syllogism	$p \vee q$ $\neg q$	p
	$p \vee q$ $\neg p$	q
Hypothetical syllogism	$p \Rightarrow q$ $q \Rightarrow r$	$p \Rightarrow r$
Resolution	$p \vee r$ $q \vee \neg r$	$p \vee q$
Dilemma	$p \vee q$ $p \Rightarrow r$ $q \Rightarrow r$	r
Constructive Dilemma	$p \Rightarrow q$ $r \Rightarrow s$ $p \vee r$	$q \vee s$
Rule of contradiction	$\neg p \Rightarrow \mathit{false}$	p

Sample problems

1. Given the statements p, q, r, s and t, and the premises

   (a) $\neg p \vee q \Rightarrow r$

   (b) $s \vee \neg q$

   (c) $\neg t$

   (d) $p \Rightarrow t$

   (e) $\neg p \wedge r \Rightarrow \neg s$

   derive the conclusion $\neg q$.

   From (c) and (d), infer $\neg p$ using Modus Tollens (1)
   From (1), infer $\neg p \vee q$ using Disjunctive Addition (2)
   From (2) and (a), infer r using Modus Ponens (3)
   From (1) and (3), infer $\neg p \wedge r$ using Conjunctive Addition (4)
   From (4) and (e), infer $\neg s$ using Modus Ponens (5)
   From (5) and (b), infer $\neg q$ using Disjunctive Syllogism.

2. Given that p, q, r, s, t, u and v all represent statements, and given the following premises

   (a) $p \Rightarrow q$

   (b) $r \vee s$

   (c) $r \Rightarrow t$

   (d) $\neg q$

   (e) $u \Rightarrow v$

   (f) $s \Rightarrow p$

   draw a reasonable conclusion using the rules of inference.

   From (a) and (d), infer $\neg p$ using Modus tollens (1)
   From (f) and (1), infer $\neg s$ using Modus tollens (2)
   From (b) and (2), infer r using disjunctive syllogism (3)
   From (c) and (3), infer t using Modus ponens (4)

   So the conclusion is t.

   Notice that the premise (e) is not used in this problem.

Translating informal statements into propositions

1. Finding the hidden treasure.
   In the back of an old cupboard you discover a note signed by a pirate for his bizarre sense of humor and love for logical puzzles. In the note, he wrote that he had hidden a treasure somewhere on the property. He listed five clues for you and challenged that the readers should use these clues to find out the treasure.

   (a) If the house is next to a lake, the the treasure is not in the kitchen.

   (b) If the tree in the front yard is an elm, then the treasure is in the kitchen.

   (c) This house is next to a lake.

   (d) The tree in the front yard is an elm or the treasure is buried under the flagpole.

   (e) If the tree in the back yard is an oak, then the treasure is in the garage.

   Where is the treasure hidden?

   Solution:
   Let

   a - House is next to lake.
   b - Treasure is in the kitchen.
   c - Tree in the front yard is an elm.
   d - Treasure is buried under the flagpole.
   e - Tree in the backyard is an oak.
   f - Treasure is in the garage.

   $$a \Rightarrow \neg b \tag{1}$$
   $$c \Rightarrow b \tag{2}$$
   $$a \tag{3}$$
   $$c \vee d \tag{4}$$
   $$e \Rightarrow f \tag{5}$$

   From (3) and (1), using Modus Ponens, infer $\neg$ b (6)
   From (6) and (2), using Modus Tollens, infer $\neg$ c (7)
   From (7) and (4), using Disjunctive Syllogism, infer d (8)
   Therefore, the treasure is buried under the flagpole.

   Notice that axiom (5) is not used in this derivation.

2. Who is the thief?
   There was a theft in Mr. McGregor's shop. The detective investigating this case got the following clues about three suspects A,B and C:

(a) Each of A, B, C had been in the shop on the day of the robbery and no one else had been in the shop that day.

(b) If A is guilty, then he had exactly one accomplice.

(c) If B is innocent, so is C.

(d) If exactly two are guilty, then A is one of them.

(e) If C is innocent, so is B.

Help the detective to find the thief.

<u>Solution:</u>
Let

p	- Each of A,B, and C had been in the shop on the day of theft.
q	- No one else had been in the shop on that day.
r	- A is guilty.
s	- B is guilty.
t	- C is guilty.

It is assumed that guilty $\equiv \neg$ innocent.

$$p \tag{1}$$
$$q \tag{2}$$
$$r \Rightarrow (s \vee t) \tag{3}$$
$$\neg\ s \Rightarrow \neg\ t \tag{4}$$
$$(r \wedge s) \vee (r \wedge t) \tag{5}$$
$$\neg\ t \Rightarrow \neg\ s \tag{6}$$

There is no solution to this problem.

3. Find the mistake in a computer program.
Given the following information about a computer program, draw a reasonable conclusion that finds the mistake in the program:

(a) There is an undeclared variable or there is a syntax error in the first five lines.

(b) If there is a syntax error in the first five lines, then there is a missing semicolon or a variable name is misspelled.

(c) There is not a missing semicolon.

(d) There is not a misspelled variable name.

<u>Solution:</u>
Let

a - There is an undeclared variable.
b - There is a syntax error in the first five lines.
c - There is a missing semicolon.
d - A variable name is misspelled.

$$a \lor b \tag{1}$$
$$b \Rightarrow (c \lor d) \tag{2}$$
$$\neg\ c \tag{3}$$
$$\neg\ d \tag{4}$$

From (3) and (4), using Conjunctive Addition, infer $\neg\ c \land \neg\ d$ (5)
From (5), applying DeMorgan's law, derive $\neg\ (c \lor d)$ (6)
From (6) and (2), using Modus Tollens, infer $\neg\ b$ (7)
From (7) and (1), using Disjunctive Syllogism, infer a (8)

Therefore, there is an undeclared variable.

Necessary and Sufficient Conditions

The notes for this section is taken from the book

Jon Barwise and John Etchemendy, *Language, Proof and Logic*, CSLI Publications, Stanford University, 2008, ISBN: 978-1-57586-374-0.

if Clause
In English, the *if* clause introduces a *sufficient condition.* It is often translated into an implication. For example, the statement

If I get one more quarter, I can buy a coffee

is translated into the formal statement $P \Rightarrow Q$ where P stands for the expression "I get one more quarter" and Q stands for the expression "I can buy a coffee". In this case, it is *sufficient* to satisfy P in order to evaluate Q.

only if Clause
The clause *only if* introduces a *necessary condition* which is stronger than a sufficient condition. However, it may be incorrectly translated into a formal statement. As an example, consider the statement

I can buy a coffee only if I get one more quarter

where P refers to "I get one more quarter" and Q refers to "I can buy a coffee". If observed closely, the statement makes a stronger assertion that getting one more quarter is more important in order to buy a coffee. Stated otherwise, getting one more quarter is a *necessary* condition to buy a coffee. Therefore, the same statement can be rewritten as

If I do not get one more quarter, I cannot buy a coffee

Translating this statement, we will get $\neg P \Rightarrow \neg Q$ which is the same as $Q \Rightarrow P$ (using Modus Tollens).

unless Clause

Like *only if*, the clause *unless* is also misunderstood during translation. Let us look at the following example:

I will graduate unless I fail in CS 743

The intention of the statement is that I need to pass the course CS 743 in order to graduate. So, it can be stated as

If I do not fail in CS 743, I will graduate

which is formally translated into $\neg P \Rightarrow Q$ where P stands for "I fail in CS 743" and Q stands for "I will graduate". A close observation of this discussion gives a clue to rewrite the term *unless* by *if ... not*. So, the above statement is equivalent to

I will graduate if I do not fail in CS 743

Thus, whenever a statement of the form "P unless Q" is given, it should be formally translated into $\neg Q \Rightarrow P$.

except clause

One more word that is often misunderstood during the translation process is *except*. This word makes the conclusion of otherwise a normal situation into a different path just for one case. Consider the statement

I am able to read without glasses except when I read fine-prints

This sentence indicates that the person normally does not require glasses for reading but when the document has fine-prints, this person requires glasses. A similar situation occurs in computer software where everything seems to be correct *except* for some specific cases. When the word *except* occurs in a statement such as "P except Q", it is translated into "if Q, then NOT P. So the previous sentence will be translated into

If I read fine-prints then NOT (I am able to read without glasses).

Quick Review

Original statement	Translation
if P then Q	
P only if Q	if $\neg$ Q then $\neg$ P
P unless Q	P if $\neg$ Q
P except Q	if Q then $\neg$ P

<u>Another Example</u> The following example is given to show that sometimes it is not possible to draw a reasonable conclusion either because there is not sufficient number of hypotheses given or there is a mistake in the translation process.

1. Completing an assignment.
   Given the following information, draw a reasonable conclusion:

   (a) If a student takes the Software Engineering course, he/she must have taken the Database course.

   (b) A student taking the Software Engineering course can develop a GUI.

   (c) The assignment can be completed by solving the database problem and developing a GUI.

   (d) A database problem can be solved by a student only if he/she has taken the Database course.

   If you are not able to come to a reasonable conclusion, what changes can be made to the statements in order to arrive at a reasonable conclusion?

   <u>Solution:</u>
   Let

   a - Students takes Software Engineering course.
   b - Student has taken Database course.
   c - Student can develop a GUI.
   d - Assignment can be completed.
   e - Solve database problem.

   $$a \Rightarrow b \tag{1}$$
   $$a \Rightarrow c \tag{2}$$
   $$e \wedge c \Rightarrow d \tag{3}$$
   $$\neg\ b \Rightarrow \neg\ e \tag{4}$$

   This set of axioms does not lead to any reasonable conclusion. If the statement "A database problem can be solved by a student only if he/she has taken the Database course" is revised into "If a student has taken the Database course, he/she can solve a database problem", then axiom (4) becomes $b \Rightarrow e$. Using the new axiom (4), the following conclusions can be derived:

From (1) and new (4), using hypothetical syllogism, infer a $\Rightarrow$ e (5)
From (2) and (5), using conjunctive addition, infer (a $\Rightarrow$ c) $\wedge$ (a $\Rightarrow$ e) 6)
Axiom (6) can be simplified using distributive law as a $\Rightarrow$ (c $\wedge$ e) (7)
From (7) and (3), using hypothetical syllogism, infer a $\Rightarrow$ d (8)

Therefore, if a student takes the Software Engineering course, he/she can complete the assignment.