Abstract

We examine the structural role of negation in a logical system. We show how a preservationist approach to consequence relations can illuminate this role. A general account of consequence is given, in terms of the preservation of any among many candidate properties of sets of sentences in a logical language. Such properties are defined, and important features are described. We show that it can be difficult to define a “negation” in the resulting logical system. We then give a new account of same, using the idea of logical opposition. We prove that every sentence in a logical language has some logically opposed set, and argue that such a set can be regarded as doing the structural work of negation.

1 Introduction

This paper examines the structural role of negation in systems of logic through the lens of the preservationist approach, which takes a broader view of what logical consequence can be. As a form of logical pluralism, preservationism does not generally intend to redefine truth, nor consistency, nor any of the classical notions. Instead, it simply extends some eminently classical ideas to wider contexts. In these contexts, rather than focusing exclusively on the traditional goal of preserving truth or satisfiability between premisses and conclusions, we can look at any number of other properties that we might care about. If we find any such property of premiss-sets valuable and interesting enough, and if it is not naturally monotonic—so that it is not simply preserved under any old additions—then we may seek a system of consequence that retains the property. Here, we are not dismissing the importance or validity of classical notions. Rather, we are simply adding other potentially worthy set properties to the mix, asking how me might keep them around as well.

Our approach to preservationism here is highly general. We consider a wide range of possible properties that characterize sets of sentences in a logical language. This collection of properties, which we will define precisely, we call Lindenbaum properties; in essence, any property is Lindenbaum just in case a set of sentences has the property if and only if it can be extended to a possibly larger set that is maximal with respect to the property. Classical consistency
and satisfiability are among the Lindenbaum properties, but there are indefinitely many more, each distinct, and each potentially useful. Given any such property, it is straightforward to define a consequence-relation that preserves it (although defining rules of inference may be a wholly different matter). Such a relation, it turns out, has many of the structural features usually associated with consequence.

There is, however, one potential difficulty with any such account, centered around the issue of negation and its role in logical reasoning. In many cases, it turns out, the system of consequence we devise does not behave in the usual way with respect to sentence-level negation. That is, it may no longer be the case, for instance, that a classically inconsistent set containing both some literal $p$ and its negation $\neg p$ is unprincipled; that is, our logic need not generate all possible consequences from such a set, since the set can still have our new Lindenbaum property, even if it lacks classical consistency. This can lead to potential complications when it comes to reasoning, since we can no longer rely on some of the traditional bivalence assumptions useful in classical contexts.

Previous authors have tried to solve this problem by considering logics that feature some form of denial, so that even if the functional negation of $p$ does not cause a loss of logical composure, there is some sentence $\text{some sentence}$ that can be added to any set containing $p$ that will do so. As we will explain, however, many perfectly good preservationist systems, for genuinely interesting sentence-set properties, simply lack the requisite denial. Furthermore, even where they do have it, it is not guaranteed to do all the work that negation does in classical settings. We therefore present a new idea, logical opposition, which turns out to do the trick. For each sentence, we will show, there exists some associated set of sentences that behaves exactly in the way that we would like, returning to us many of the structural properties associated with negation.

On the one hand, this proof of guaranteed existence already distinguishes logical opposition from logical denial. On the other hand, we will argue that these results are not merely a matter of formal interest, but reveal something about what it is for a logical entity to count as “a negation” at all.

1.1 Structure of the Paper

We begin our paper, in Section 2, by outlining the basic ideas behind preservationism in logic. Then, in Section 3, we define what it is for a set property to be Lindenbaum, and consider the notion of maximality with respect to such a property. For any such property, we show how to define a consequence relation that preserves it, and demonstrate some of the structural features any such relation will have. Finally, we consider some examples of Lindenbaum properties, and show how indefinitely many of them can be generated. Section 4 contains our main results. After considering some important features of classical logical negation, we discuss previous attempts to isolate a “negation”-like object, and show why such attempts do not generally solve our problem. We then define a new notion, logical opposition, that
succeeds; for every sentence, it is proved, there exists a logically opposed set, relative to any Lindenbaum property. Furthermore, we show, such a set behaves like classical negation, providing us with newly valid versions of the bivalence properties of classical consequence systems. Section 5 discusses our results, and draws some conclusions. Finally, we note that to make the paper more readable, some longer proofs are not included in the main text, but follow in a separate Formal Appendix, appearing just before the references.

1.2 Notation and Other Conventions

Throughout this paper, we will be dealing with the properties of sets of sentences, taken from some logical language \( L \). We will presume that language \( L \) is countable, and that it is straightforwardly denumerable, so that we can sensibly talk of putting it into some fixed order; beyond that, we do not usually assume much more about it. We shall use upper-case greek letters, \( \Sigma, \Gamma, \Delta, \ldots \), to represent sets of sentences from \( L \), and lower-case greek, \( \alpha, \beta, \delta, \ldots \), for individual such sentences. One exception, which we trust will not be too confusing, will be the use of \( \varphi \) as a predicate of sentence-sets; as we explain below, we will use for example \( \varphi(\Sigma) \) to mean that the set \( \Sigma \) has the property \( \varphi \), and \( \overline{\varphi}(\Sigma) \) to mean that it does not.

We will be discussing a range of defined consequence and provability relations, given relative to some set-property \( \varphi \), and when we do so, we will use subscripted notation, such as \( \models_{\varphi} \) and \( \vdash_{\varphi} \). When symbols \( \models \) and \( \vdash \) appear without the subscript, we will always mean them to stand for classical entailment and provability relations. Similarly, any use of the true and the false, \( \top \) and \( \bot \), without subscripts refers to classical theorems and absurdities respectively; where necessary, we use subscripted versions of these, too.

Basic set-theoretical notation will be used. One convention is the use of commas for union of sets where singletons are concerned; that is, we will sometimes write \( (\Sigma, \alpha) \) for the union \( (\Sigma \cup \{\alpha\}) \).

2 Preservationism and Consequence

The preservationist approach to logic can be traced to work first developed in the early 1980s by Raymond E. Jennings and Peter K. Schotch, who introduced the foundational notion of level-preservation.\(^1\) As the story is now usually told, the general approach is tied to a generalization of the classical understanding of logical consequence as the preservation of truth—or more precisely, the preservation of classical satisfiability and consistency. That is, we can define the consequences of a set of sentences \( \Sigma \) in terms of its satisfiable or consistent extensions:

\(^1\)Relevant works are [Jennings and Schotch, 1980; 1984; Schotch and Jennings, 1989]. Brown [2007] gives a detailed history of developments in the area, placing these earlier papers in wider context.
1. $\Sigma \vdash \alpha$ iff $\forall \Sigma'[\Sigma' \supseteq \Sigma \& \Sigma'$ is satisfiable] $\Rightarrow (\Sigma', \alpha)$ is satisfiable.

2. $\Sigma \vdash \alpha$ iff $\forall \Sigma'[\Sigma' \supseteq \Sigma \& \Sigma'$ is consistent] $\Rightarrow (\Sigma', \alpha)$ is consistent.

In brief, the preservationist impulse arises as soon as one observes this fact and then considers the possibility of replacing “satisfiable” or “consistent” in these formulations with some other property of sentence sets.

Much of the associated work has looked specifically at properties which hold of classically inconsistent sets, seeking principled inference even in the presence of overall unsatisfiability; that is, the logics that have been developed are often paraconsistent. In classical logic, of course, all sentences $\alpha$ follow from any inconsistent set $\Sigma \vdash \bot$, since the extensional definitions just given are vacuously satisfied. While there are a variety of distinct approaches to paraconsistency, they all generally avoid this presumption: in paraconsistent systems, the consequences of classically inconsistent sets do not “explode” to include just anything whatsoever. In its preservationist version, this approach involves finding ways of measuring the degree to which a set is classically inconsistent, and deriving consequence relations that preserve such a measure where possible.

It is important to note, however, that nothing demands that the preservationist approach lead to paraconsistency. In general, when we speak of finding other properties—something other than classical consistency or satisfiability—to preserve in sentence sets, we do not necessarily presume that our consequence relations must be principled with respect to inconsistency. Rather than seeking to broaden the range of sets from we can draw reasonable inference, we might seek to restrict it; that is, we may look for other properties in addition to the classical ones, that our consequence relations must preserve. For instance, in an epistemic context we may want our logical system not only to yield consistent extensions of our consistent beliefs, but also to preserve some degree of warrant, only allowing principled inference from sets of beliefs about which we possess some measure of certainty.

We do not pursue this suggestion any further here. Rather, we simply note the fact that preservationist logic is not always about dealing with classical inconsistency. Indeed, in this paper, we will take a very general approach to the sorts of set properties that might be preserved by a consequence relation. While we will consider some particular examples, we are interested in the broader picture, namely the overall structure of a preservationist system with respect to any of a very wide range of related set properties. As we will show, any system of consequence defined relative to such a property will share some very basic features. In particular, we will argue that once we have identified our property of interest, and given a consequence relation in terms of the preservation of that property, we can then isolate a logical construct that plays the structural role of a negation, relative to any sentence of our language. What we mean by this idea—the “structural role of a negation”—will be made clear as we go along.

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2Or at least, some sets that are classically inconsistent are not explosive. There is nothing wrong, from a paraconsistent point of view, with sorting out different ways a set can be classically inconsistent, some of which permit principled inference, and some of which remain volatile. See [Priest et al., 1989] for one survey of different approaches to paraconsistency.
3 Isolating a Range of Set Properties

Sets of sentences from a logical language can have any number of properties. As already mentioned, they can be classically satisfiable or consistent; they can also be inconsistent, but only to a certain degree. Sets can also have far more prosaic properties: they can be non-empty, or they can contain exactly \( n \) members, for instance. Some such set properties have proven interesting to logicians, and some have not, in the sense that we have not generally tried to devise consequence relations for those properties that we find uninteresting. While much work has been devoted to systems that preserve classical consistency, there has been no real interest in consequence relations that preserve non-emptiness. In the latter case, of course, seemingly anything works: adding any sentence whatsoever to a set yields a non-empty extension, and so the overall consequence relation would be equally unprincipled (or perhaps vacuously principled) from all sets. It is indeed hard to see how such a consequence relation could correspond to any useful system of reasoning from premisses to conclusions.

We would, however, like to say more than that some properties “seem uninteresting” or that some consequence relations “do not seem useful.” Rather, we seek to isolate some particular characteristics that set properties can have, identifying properties of properties that themselves lead to interesting results. We begin by defining two such characteristics, and drawing out some of their consequences. Later, we will provide some examples of set properties that do, or do not, have these characteristics. Our argument is not that the following definitions identify everything about sentence sets that might conceivably be of logical interest. Rather, we simply mean to show that if our sets have the following sorts of properties, then something interesting follows from that fact.

**Definition 1** (Genuine). A property \( \varphi \) of sets of sentences \( \Sigma \subseteq L \) is genuine if and only if it is:

1. **Well-defined**: \( \forall \Sigma [\varphi(\Sigma) \text{ or } \neg \varphi(\Sigma)] \) and \( \neg \exists \Sigma [\varphi(\Sigma) \text{ and } \neg \varphi(\Sigma)] \).
2. **Non-vacuous**: \( \exists \Sigma \varphi(\Sigma) \).
3. **Non-trivial**: \( \exists \Sigma \neg \varphi(\Sigma) \).

That is, we are interested broadly in properties that are either applicable or not, exclusively, to every set, while being neither too broad nor too narrow. Beyond this general range of properties, we shall also be interested in those with a more significant characteristic, however.

**Definition 2** (Compact). A property \( \varphi \) of sets of sentences \( \Sigma \subseteq L \) is compact if and only if:

\[
\forall \Sigma [\varphi(\Sigma) \iff \forall \Sigma' ((\Sigma' \subseteq \Sigma \text{ and } \Sigma' \text{ is finite}) \Rightarrow \varphi(\Sigma'))]
\]
Compactness clearly isolates a much narrower range of properties than does our first definition; not every well-defined property of sentence sets is preserved by all finite subsets, nor does the fact that every finite subset has a property guarantee that the superset will have the same property.\(^3\) However, as we shall see, this second characteristic still allows us to pick out a very wide range of interesting qualities that a set of sentences might have. Furthermore, it is a long-established and important fact that any set possessed of a compact property \(\varphi\) can be extended to maximality, in the following sense.

**Definition 3** (Maximal-\(\varphi\) set). \(\Sigma^\ast \subseteq L\) is a **maximal-\(\varphi\)** set if and only if:

1. \(\varphi(\Sigma^\ast)\).
2. \(\forall \alpha [\alpha \notin \Sigma^\ast \Rightarrow \varphi(\Sigma^\ast)]\).

For any set \(\Sigma\), we say that \(\Sigma^\ast\) is a **maximal-\(\varphi\)** extension of \(\Sigma\) if and only if \(\Sigma \subseteq \Sigma^\ast\) and \(\Sigma^\ast\) is a maximal-\(\varphi\) set. Finally, \([\Sigma]\_\varphi^\ast\) is defined as the set of all maximal-\(\varphi\) extensions of \(\Sigma\).

Without the stipulation that a property \(\varphi\) is compact, we cannot be sure that every set with that property has any maximal-\(\varphi\) extensions. Of course, if \(\varphi(\Sigma)\), then \(\Sigma\) has **some superset** with property \(\varphi\), since every set is a subset of itself. Unfortunately, absent compactness we can not be sure that the maximality condition is guaranteed, and there may not be a well-defined superset \(\Sigma^\ast\) with property \(\varphi\) such that the addition of any formula \(\alpha\) not already in it causes the loss of that property. Where compactness is present, however, the existence of maximal extensions is guaranteed, a fact originally proven by John Tukey [1940], and independently by Teichmüller.\(^4\)

**Lemma 1** (Tukey-Teichmüller). For any compact property \(\varphi\) and set \(\Sigma\) such that \(\varphi(\Sigma)\), \(\Sigma\) has some maximal-\(\varphi\) extension, \(\Sigma^\ast\).

**Proof.** Please see the Formal Appendix at the end of this paper.

The reader will almost certainly have recognized this lemma as a generalized version of Lindenbaum’s Lemma, the classical logical result that any consistent set has a maximal-consistent extension. Indeed, Lindenbaum’s Lemma follows directly from the Tukey-Teichmüller result, based simply on the fact that classical consistency (and classical satisfiability) is a compact property. This more general form of the lemma shows that we need not limit our interest to consistency: as we will see, many of the usual characteristics of classical consequence relations carry over in

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\(^3\)Obviously, if \(\Sigma\) is itself finite, then \(\varphi(\Sigma)\) follows from the fact that every finite subset has that property, since \(\Sigma \subseteq \Sigma\). However, there will be cases involving infinite sets \(\Sigma\) where this does not immediately follow.

\(^4\)To be absolutely precise, we should note that in Tukey’s work, the terminology is not “compactness”; rather, he says that a property \(\varphi\) as in Definition 2 “is of finite character.” Furthermore, the discussion applies to sets composed of any objects whatsoever, not simply sets of sentences from some logical language.
some form, if those relations are defined using the notion of maximal extensions relative to any genuine and compact property $\varphi$.

### 3.1 Lindenbaum Properties

Strictly speaking, Lemma 1 can apply to even vacuous or trivial properties of sets of sentences. If no set has the property $\varphi$, then it is vacuous that the property is compact, and every set $\Sigma$ such that $\varphi(\Sigma)$ has a maximal-$\varphi$ extension. Similarly, if every set has $\varphi$, then it is trivially compact, and every set has a maximal-$\varphi$ extension, namely $\Sigma^* = L$, the set of all sentences in the language. To limit ourselves, we will restrict ourselves to a set of genuine properties, and define them somewhat differently, *in terms of* their maximal-$\varphi$ extensions. Since we are operating in a logical context, we name them for Lindenbaum, of the famous lemma just mentioned.

**Definition 4** (Lindenbaum Property). A property $\varphi$ of sets of sentences $\Sigma \subseteq L$ is a *Lindenbaum property* (alternatively, $\varphi$ is Lindenbaum) if and only if:

1. $\varphi$ is *genuine*, in the sense of Definition 1.
2. Sets have the property just in case they have some maximal-$\varphi$ extension: $\forall \Sigma \left[ \varphi(\Sigma) \iff [\Sigma]_\varphi^* \neq \emptyset \right]$.

Lindenbaum properties are thus intrinsically tied up with maximal extensions. Any set with some Lindenbaum property $\varphi$ has a maximal-$\varphi$ extension, and any set with some maximal-$\varphi$ extension for a Lindenbaum property $\varphi$ will have that same property itself. Note that nothing is said about the compactness of Lindenbaum properties, and it is not strictly required; indeed, maximal extensibility is built into the definition, and so no version of Lemma 1 is required. However, compactness is still of concern to us here, since it will allow us to demonstrate that a property $\varphi$ is in fact Lindenbaum.

**Proposition 1.** For any property $\varphi$ of sets of sentences $\Sigma \subseteq L$, if $\varphi$ is genuine and compact, then $\varphi$ is a Lindenbaum property.

*Proof.* Consider any genuine and compact property $\varphi$. We need only show that $\forall \Sigma \left[ \varphi(\Sigma) \iff [\Sigma]_\varphi^* \neq \emptyset \right]$. The rightward direction is simply a statement of Lemma 1: any set with property $\varphi$ will have some maximal-$\varphi$ extension. For the leftward direction, consider any set $\Sigma$ such that $[\Sigma]_\varphi^* \neq \emptyset$. That is, there exists some maximal-$\varphi$ set $\Sigma^*$ such that $\Sigma \subseteq \Sigma^*$. Now every finite subset of $\Sigma$ is a finite subset of $\Sigma^*$, and so by compactness every such finite subset has property $\varphi$, and so does $\Sigma$, as required. Property $\varphi$ is therefore Lindenbaum.

Further, we can prove that maximal-$\varphi$ sets are each in some sense *unique*, with respect to a Lindenbaum property. (While this is not limited to Lindenbaum properties necessarily, it is worth noting.)
Proposition 2 (Unique Extension). For any Lindenbaum property, there exists some maximal-$\varphi$ set $\Sigma^*$, and the only maximal-$\varphi$ extension of $\Sigma^*$ is itself: $[\Sigma^*]_{\varphi} = \{\Sigma^*\}$.

Proof. Consider any Lindenbaum property. Since such a property is genuine, there exists some set $\Sigma$ such that $\varphi(\Sigma)$, and so there exists some maximal-$\varphi$ extension $\Sigma^* \in [\Sigma]^*_{\varphi}$. Now, since $\Sigma^*$ has Lindenbaum property $\varphi$, it has a maximal-$\varphi$ extension, and so $[\Sigma^*]_{\varphi} \neq \emptyset$. Consider any $\Delta \in [\Sigma^*]_{\varphi}$, and suppose for purposes of reductio, that $\Delta \neq \Sigma^*$. Then, since $\Delta \supseteq \Sigma^*$, there exists some formula $\alpha$ such that $\alpha \in \Delta$ but $\sigma \notin \Sigma^*$. In that case, since $\Sigma^*$ is maximal, $\varphi(\Sigma^*, \alpha)$, and so that latter set has no maximal extensions, which is absurd. Thus $[\Sigma^*]_{\varphi} = \{\Sigma^*\}$.  

3.2 Lindenbaum-Preserving Consequence Relations

Before going on to outline some examples of Lindenbaum Properties, we consider one of their key features, namely the fact that we can define a $\varphi$-preserving consequence relation for any such property, in terms of maximal-$\varphi$ extensions. In all of what follows, it is implicit that property $\varphi$ is Lindenbaum.

Definition 5 ($\varphi$-Preserving Consequence). For any set $\Sigma \subseteq L$ and sentence $\alpha \in L$, $\alpha$ is a $\varphi$-preserving consequence of $\Sigma$ if and only if $\alpha$ is a member of every maximal-$\varphi$ extension of $\Sigma$: 

$$\Sigma \models_{\varphi} \alpha \iff \forall \Sigma^* \in [\Sigma]^*_{\varphi} [\alpha \in \Sigma^*].$$

We define the set of all $\varphi$-preserving consequences of a set as follows: $\text{Cn}_{\varphi}^\bowtie(\Sigma) = \{\alpha \mid \Sigma \models_{\varphi} \alpha\}$.

From this definition, another important property of maximal-$\varphi$ sets follows, namely that they contain all and only their $\varphi$-preserving consequences.

Proposition 3 (Deductive Closure). For any maximal-$\varphi$ set $\Sigma^*$, and any sentence $\alpha$, $(\Sigma^* \models_{\varphi} \alpha \iff \alpha \in \Sigma^*)$.

Proof. For the rightward direction, consider any $\alpha$ such that $\Sigma^* \models_{\varphi} \alpha$. By definition, this means that $\alpha$ is an element of any maximal-$\varphi$ extension of $\Sigma^*$. Further, by Proposition 2, $\Sigma^*$ is its own unique maximal-$\varphi$ extension, and so $\alpha \in \Sigma^*$.

For the leftward direction, consider any $\alpha \in \Sigma^*$. Again by Proposition 2, $\Sigma^*$ is its own unique maximal-$\varphi$ extension. Therefore, $\alpha$ is an element of all such extensions, and so $\Sigma^* \models_{\varphi} \alpha$.  

It also immediately follows that any set $\Sigma$ for which $\varphi(\Sigma)$ holds is unprincipled, in that every sentence is a consequence of such a set:

Proposition 4. For any set such that $\varphi(\Sigma)$, all sentences are $\varphi$-preserving consequences of $\Sigma$: $\text{Cn}_{\varphi}^\bowtie(\Sigma) = L$. 

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Proof. Consider any set of sentences such that $\varphi(\Sigma)$, for some Lindenbaum property $\varphi$, and arbitrary sentence $\alpha$. By the definition of a Lindenbaum property, such a set has no maximal-$\varphi$ extensions, and so Definition 5 is vacuously satisfied for $\alpha$. Therefore, $\forall \alpha \in L, \Sigma \vdash_\varphi \alpha$.

Thus, just as classically inconsistent sets are explosive with respect to classical consequence, the consequences of any set without some Lindenbaum property $\varphi$ explode upwards to include all possible sentences. This, however, is not the most important feature of $\varphi$-preserving consequence. Previously (Section 2), we defined classical consequence in terms of preservation of consistency over all consistent supersets of a set $\Sigma$. It is straightforward to show that $\varphi$-preserving consequences have the relevantly similar property.

**Proposition 5.** For any $\Sigma$ and $\alpha$, $\Sigma \vdash_\varphi \alpha$ if and only if: $\forall \Sigma'[(\Sigma' \supseteq \Sigma \land \varphi(\Sigma')) \Rightarrow \varphi(\Sigma', \alpha)]$.

Proof. Please see the Formal Appendix at the end of this paper.

By convention, we can write $\top_\varphi$ for any sentence $\alpha$ such that $\Sigma \vdash_\varphi \alpha$, for all sets $\Sigma$. Similarly, we can write $\bot_\varphi$ for any sentence $\beta$ such that $\varphi(\{\beta\})$. The reader can easily ascertain that the usual facts about these “true” and “false” constants hold, and they are offered without proof.

**Fact 1.** The following hold of our $\top_\varphi$ and $\bot_\varphi$ sentences:

1. $\forall \Sigma [\varphi(\Sigma) \iff \varphi(\Sigma \cup \{\top_\varphi\})]$.
2. For any maximal-$\varphi$ set $\Sigma^*$, $\top_\varphi \in \Sigma^*$.
3. $\forall \Sigma [\varphi(\Sigma) \iff \Sigma \vdash_\varphi \bot_\varphi]$.

Finally, we note that the $\varphi$-preserving relation defined in terms of maximal-$\varphi$ extensions has many of the properties usually associated with logical consequence. This shows, we argue, that the sort of relation defined here is a sensible one, in that it coheres with many of the intuitive notions about consequence.

**Proposition 6 (Structure of $\vdash_\varphi$).** The consequence relation $\vdash_\varphi$ has the following structural features (each given in terms of the consequence-set $\text{Cn}_\varphi^\Sigma(\Sigma)$):

1. **Reflexivity:** $\forall \Sigma [\Sigma \subseteq \text{Cn}_\varphi^\Sigma(\Sigma)]$.
2. **Monotonicity:** $\forall \Sigma, \Delta [\Sigma \subseteq \Delta \Rightarrow \text{Cn}_\varphi^\Sigma(\Sigma) \subseteq \text{Cn}_\varphi^\Delta(\Delta)]$.
3. **Idempotence:** $\forall \Sigma [\text{Cn}_\varphi^\Sigma(\text{Cn}_\varphi^\Sigma(\Sigma)) = \text{Cn}_\varphi^\Sigma(\Sigma)]$.
4. **Transitivity:** $\forall \Sigma, \Delta [\Delta \subseteq \text{Cn}_\varphi^\Sigma(\Sigma) \Rightarrow \text{Cn}_\varphi^\Sigma(\Delta) \subseteq \text{Cn}_\varphi^\Delta(\Sigma)]$.

Proof. For a detailed proof, see the Formal Appendix at the end of the paper.
3.2.1 A Note on Monotonicity

It is important to stress that these properties of a \( \varphi \)-preserving relation are not meant to be conclusive evidence that it “is consequence outright.” We simply argue that these features, since they are shared by classical and other well-known consequence relations, mean that calling our relation a “consequence” relation is not outrageous. Nothing about this claim is meant to be definitional, and we should note that many preservationist systems do not feature it. In particular, it will often be the case that such a system of consequence will lack monotonicity, or feature it only in a weakened form. In such systems, the formation of supersets of some set can in fact create sets lacking properties of the original, but without licensing explosively unprincipled inference; indeed, in these approaches we can lose the ability to draw certain consequences when we move up to supersets of our original premisses.

Such a feature is present in the forcing systems of Jennings and Schotch, to which we have already referred. Designed for reasoning with sets that can have various finite levels of incoherence, forcing “floats” with the level of a set, so that the addition of more inconsistency when forming supersets leads to the use of weaker forms of inference, preserving principled consequence at the price of monotonicity. In such systems, of course, weaker forms of monotonicity can replace the full form that we have here. We are not interested in defending one approach over the other: no claim about whether a system requires full monotonicity to be a system of consequence is intended, and we simply note the difference. Brown and Schotch [1999] and Payette and Schotch [2009] contain some interesting discussion of the monotonicity matter, if the reader is interested in pursuing the question further.

3.3 Some Examples of Lindenbaum Properties

So far, this has been presented in highly abstract form. As it turns out, however, many different properties are Lindenbaum. Some such properties have been well-studied, while others are relatively unknown. Of course, on the one hand, the classical consistency (or satisfiability) property is Lindenbaum, since it is clearly genuine, and all sets of sentences are classically consistent just in case they have maximal-consistent extensions. This is only one such possibility, however: there are many other distinct Lindenbaum properties of sentence-sets, some of which are quite interesting from a preservationist point of view.

One such set of properties involves the notion of level. As we have mentioned, this notion was first introduced in paraconsistent settings in terms of levels of classical consistency, but following the lead of Payette and d’Entremont [2006], among others, we define it in more general form here. In presenting the idea, we rely upon the existence of two things, each drawn from some single underlying logical system. First, we require some logical consequence relation \( \models_X \), with corresponding consequence-set \( Cn^p_X(\Sigma) \) for any set \( \Sigma \). Second, we require a consistency
property $\varphi$ that can hold of sets of sentences, where $\varphi(\Sigma)$ means that the set is “consistent” relative to $\models_X$ just in case there is some sentence $\alpha$ such that $\Sigma \not\models_X \alpha$. Given these two items, we can define a notion of “covering” for logical sets.

**Definition 6 (Logical Cover).** The indexed set of sets $\mathfrak{F} = \{\Delta_0, \Delta_1, \ldots, \Delta_\xi\}$ (where $\xi$ is some ordinal value) is a logical cover of set $\Sigma$, relative to some consequence relation $\models_X$ and associated consistency property $\varphi$—written $\text{Cov}_\varphi[\mathfrak{F}, \Sigma]$—if and only if:

1. $\Delta = \emptyset$.
2. $\forall i \leq \xi, \varphi(\Delta_i)$.
3. $\Sigma \subseteq \bigcup_{i \leq \xi} \text{Cn}_X(\Delta_i)$.

That is, $\mathfrak{F}$ covers $\Sigma$ provided that $\mathfrak{F}$ is a collection of sets, beginning with the empty set, such that each has the consistency property $\varphi$, and they collectively have all members of $\Sigma$ among their consequences. We can speak of the width of a cover to be its cardinality, less one: $w(\mathfrak{F}) = |\mathfrak{F}| - 1$. Doing so then allows us to define the logical level of any set $\Sigma$.

**Definition 7 (Level).** The level (relative to some consequence relation $\models_X$) of any set $\Sigma$ is the width of its narrowest logical cover, if one exists:

$$\ell^\varphi_X(\Sigma) = \begin{cases} 
\min_{w(\mathfrak{F})} \{ w(\mathfrak{F}) \mid \text{Cov}_\varphi[\mathfrak{F}, \Sigma] \} & \text{if such a cover and limit exists;} \\
\infty & \text{else.} 
\end{cases}$$

Thus, for instance, if we take our property $\varphi$ to be classical consistency, and our consequence relation $\models_{CL}$ to be classical consequence, the level of a set $\Sigma$ corresponds to the narrowest collection of consistent sets such that the collection as a whole classically entails all elements of $\Sigma$. It thus follows obviously that $\ell^\varphi_{CL}(\Sigma) = 0$ whenever $\Sigma$ consists only of classical tautologies, while $\ell^\varphi_{CL}(\Sigma) = 1$ whenever the set is classically consistent, but contains some non-tautological sentences. Furthermore, any set $\Sigma$ that can be divided up into any countable number of consistent subparts will have some $\ell^\varphi_{CL}(\Sigma) = m \leq \omega$. Finally, any set $\Sigma$ which contains some classically absurd sentence, $\bot$, will have $\ell^\varphi_{CL}(\Sigma) = \infty$, since there exists no collection whatsoever of classically consistent sets that entails $\bot$. Thus, level of consistency distinguishes various degrees to which a set $\Sigma$ can be either consistent or inconsistent.

These sorts of level properties have been studied extensively, and many interesting results have been proven for them. However, it is important to note that logical level, given as a fixed value, is not a Lindenbaum property even if any Lindenbaum property is a consistency property relative to $\varphi$-preserving consequence, $\models_{\varphi}$, by definition.
where we are careful to ensure that it is genuine and holds of some but not all sets. For instance, consider a consistent set, consisting of one contingent and one tautologous propositional logical sentence, such as \( \Sigma = \{ p, q \lor \neg q \} \).

As we have noted, for such a set \( \ell_X(\Sigma) = 1 \), relative to classical consequence and consistency. However, if we define our property of interest to be \( \varphi(\Sigma) := (\ell_X(\Sigma) = 1) \), the resulting \( \varphi \) is not Lindenbaum, since now we have maximal-consistent extensions \( \Sigma \in [\Sigma]_{\varphi}^* \) for which \( \varphi(\Sigma^*) \) as well (they will also have level 1), but which have subsets \( \Sigma' = \{ q \lor \neg q \} \) such that \( \varphi(\Sigma^*) \), since for those subsets \( \ell_X(\Sigma') = 0 \).

This is easily remedied, however, by moving from a particular level \( m \), to include all levels less than or equal to \( m \). That is, if we simply set property \( \varphi(\Sigma) := (\ell_X(\Sigma) \leq n < \omega) \), we have a Lindenbaum property once again. It is a fundamental result of d’Entremont [1982] that for any compact property—classical consistency is a fine example—a level measure defined in terms of such a downward ordering is itself compact.

**Theorem 1** (Level Compactness [d’Entremont, 1982]). If \( \Sigma \) is a set of sentences with \( \ell_X(\Sigma) < \omega \) relative to a compact property \( \varphi \), then \( \ell_X(\Sigma) \leq n \) if and only if every finite subset \( \Sigma' \subseteq \Sigma \) has \( \ell_X(\Sigma') \leq n \).

**Proof.** For full details of the proof, see [d’Entremont, 1982; Payette and d’Entremont, 2006].

As we have already pointed out, any genuine and compact property will be Lindenbaum (Prop. 1). We now have that any property \( \varphi(\Sigma) := (\ell_X(\Sigma) \leq n < \omega) \) will be Lindenbaum so long as it is genuine, and so long as \( \ell_X(\Sigma) \) is itself defined in terms of a compact property \( \varphi' \). Level of consistency is then just one example. Indeed, we can iterate the construction of properties effectively endlessly: since level of consistency is itself compact, d’Entremont’s theorem assures us that level of level of consistency is also compact. We can always let \( \varphi(\Sigma) := (\ell_{CL}(\Sigma) \leq n < \omega) \) be our consistency property, and let our consequence-relation used in Definitions 6 and 7 be \( \models_{\varphi} \). By doing so, since the original \( \ell_{CL}(\Sigma) \) property is compact (by the compactness of classical consistency), we can now define a new Lindenbaum property, \( \varphi'(\Sigma) := (\ell_{\varphi}(\Sigma) \leq m < \omega) \), which holds of \( \Sigma \) just in case there is some covering collection of width \( \leq m \), where (1) each \( \Delta_i \) in the collection has level of consistency \( \ell_{CL}(\Delta_i) \leq n \), and (2) the collection generates all of \( \Sigma \) under the consequence relation \( \models_{\varphi} \). Thus, for our new Lindenbaum property \( \varphi' \), we can be assured of extensibility to \( \varphi' \)-preserving extensions, and we can define a new consequence relation, \( \models_{\varphi'} \) in those terms.

If we wanted to, we could then iterate the process again, using our new property \( \varphi' \), defining level of level of consistency. Indeed, nothing stops us from continuing the process as many times as we would like. Thus, there are indefinitely many distinct level properties, based solely on classical consistency at the root. Furthermore, we could begin with other interesting measures, and proceed iteratively from there, generating ever more Lindenbaum properties, and defining consequence relative to them.\(^6\)

\(^6\)Other possibilities, for instance, might be based upon something like dilution of inconsistency, where we begin by measuring the dilution of a
this. It remains the case, however, that we can do this. By taking the preservationist point of view, we open up a wide range of properties for consideration. If we find one interesting, or useful, for purposes of some task to do with reasoning from sentence-sets, then we can define a consequence relation that preserves that property. Furthermore, as we shall now show, any such theory of consequence will share a key feature: each will contain a logical item analogous to classical negation.

4 Negation and Preservation

In order to explain what we mean by this last claim, we need to consider how classical negation may or may not function in a preservationist context. On the one hand, of course, the usual negation of any sentence $\alpha$ may also be part of our language $L$. Indeed, if we base our preservationist study on any typical logical language, this will most certainly be the case. However, we shall argue, the mere presence of negation in terms of sentences that use some unary connective called “negation,” or in terms of the equivalence class of sentences that are truth-functionally identical to the classical negation of any sentence, is not enough. There are important things that the negation of a sentence $\alpha$ does structurally in classical logic, which that same sentence will not do in preservationist contexts. As we will show, the notion of what “negation” is needs to be expanded in a preservationist context, taking account of these structural features.

4.1 A Structural Account of Classical Negation

We have already shown that for any Lindenbaum property $\varphi$, any maximal-$\varphi$ set $\Sigma^*$ is its own unique extension, and is deductively closed (Propositions 2, 3). These properties are of course well-known for classically maximal-consistent sets. However, the latter also have an important property that is not shared by general Lindenbaum maximal-$\varphi$ sets: they are strongly bivalent, in the sense that for any sentence $\alpha$, and any maximal-$\varphi$ set $\Sigma^*$, $\alpha \in \Sigma^*$ if and only if $\neg \alpha \notin \Sigma^*$. That is, every classically maximally-consistent set contains exactly one “representative” of any sentence: either $\alpha$ or its negation appears, and never both.

This property of classical maximal-consistency is useful in many contexts, but it is easy to see that it fails to hold for many Lindenbaum properties $\varphi$. Take the basic level of consistency property for instance, where $\varphi(\Sigma) := c_{CL}(\Sigma) \leq 2$, all based on classical consistency and consequence. In such cases, the set of a single literal $\Sigma = \{p\}$ has the property $\varphi$, since $c_{CL}(\Sigma) = 1$; however, it is also the case that the expanded set containing $p$’s negation, $\Sigma' = \{p, \neg p\}$ also set, $d(\Sigma)$ in terms of its smallest incoherent subset (and setting dilution of any consistent set to be $\infty$). In such a case, it is relatively easy to show that $\varphi(\Sigma) := (d(\Sigma) \geq m > 0)$ is a compact, Lindenbaum property. This would allow us to move on to multiple iterations of level of dilution, or even consider dilution of level.
has \( \varphi \), since \( \ell \models \mathcal{CL}(\Sigma') = 2 \). Thus, since \( \varphi \) is Lindenbaum, we know that it has some maximal-\( \varphi \) extension \( \Sigma^* \in \lceil \Sigma' \rceil^*_\varphi \), and so there exists some maximal-\( \varphi \) set containing both \( p \) and its negation \( \neg p \), contrary to strong bivalence.

This failure leads directly to another. In classical logic, there is a tight connection between the ability to derive a sentence’s negation and inconsistency:

\[
\forall \Sigma, \alpha \left( (\Sigma, \alpha) \models \bot \iff \Sigma \models \neg \alpha \right).
\]

That is, the addition of \( \alpha \) to \( \Sigma \) results in an inconsistent set if and only if that set already entails the negation of \( \alpha \) from \( \Sigma \) (and the same holds, obviously, for the provability relation, \( \vdash \)). This is an important logical fact, key to many proofs and claims in classical logic, but it is again easy to see that it fails to hold for general Lindenbaum properties \( \varphi \). The example just given, of a maximal-\( \varphi \) set \( \Sigma^* \) that contains both \( \alpha \) and its negation, suffices; since any such set is deductively closed, \( \Sigma^* \models \neg \alpha \), but \( \Sigma^* \cup \{ \alpha \} = \Sigma^* \), and so \( \varphi(\Sigma^*, \alpha) \). Thus, we can not use the presence of a sentence’s negation to conclude that a set no longer has our property, nor can we be sure that our maximal-\( \varphi \) extensions will keep negation out of the picture. As we shall see, however, this is not the end of the story.

### 4.2 Logical Denial

One possible candidate for the logical role of negation is what Payette and others have called denial [Payette and d’Entremont, 2006; Payette and Schotch, 2009]; while they are concerned with particular consistency predicates, we define it here in terms of any Lindenbaum property.

**Definition 8** (Denial). Relative to a Lindenbaum property \( \varphi \), a logical language \( L \) has denial if and only if, for every sentence \( \alpha \in L \), there exists some other sentence \( \beta \in L \) such that \( \varphi(\{ \alpha, \beta \}) \). Language \( L \) has non-trivial denial if and only if there is some such \( \beta \) that is not an absurdity, so that \( \varphi(\{ \alpha, \beta \}) \) but \( \varphi(\{ \beta \}) \).

This idea is clear, but two things are worth noting, however. First, for many Lindenbaum properties, there will be no non-trivial denial; for instance, for the level of consistency property \( \varphi(\Sigma) := \ell_{\mathcal{CL}}^L(\Sigma) \leq 3 \), and any contingent, non-absurd sentence \( \alpha \), there exists no single non-absurd sentence \( \beta \) for which \( \ell_{\mathcal{CL}}^L(\{ \alpha, \beta \}) > 3 \) (since any such combined set can have level of consistency no greater than 2). Second, denial of the trivial kind does not give us the sort of bivalence properties we would ultimately like. To continue the previous example, consider the set \( \Sigma = \{ \neg p \land q, \neg p \land \neg q \} \). Such a set has level of consistency 2, and so \( \varphi(\Sigma) \) and there exists some maximal-\( \varphi \) extension \( \Sigma^* \in \lceil \Sigma \rceil^*_\varphi \); however, no such \( \Sigma^* \) will contain the literal expression \( p \), since then it would contain a subset \( \Sigma' = \{ p, \neg p \land q, \neg p \land \neg q \} \) for which level is 3, and property \( \varphi \) would fail to hold. Furthermore, since \( \Sigma^* \) is maximal-\( \varphi \), it
obviously does not contain, nor have as a consequence, any denial of \( p \); such a denial would be itself absurd, and so can not appear in any set with property \( \varphi \).

Where there do exist non-trivial denials, such problems can still remain. As some of the previously-cited authors describe, some of these problems can be avoided in special cases, where logics are *productival* (essentially, they have some form of conjunction operation; see [Payette and d’Entremont, 2006] for one account). Furthermore, Payette [2009] has proposed the notion of a *Negation-Denial*, which plays the role we are looking for, a non-absurd denial of any contingent sentence \( \alpha \) that behaves in accordance with the form of equation (1), and follows from any sentence that is \( \varphi \)-inconsistent with \( \alpha \). We do not follow up on these avenues, partly for reasons of space, and partly because one unavoidable fact remains: perfectly good preservationist logics often fail to have such features (indeed any system without non-trivial denial certainly will not have Negation-Denial), and so they can not be relied upon to exist.

This is a philosophically and logically unsatisfactory situation. Intuitively, it seems odd that we can talk of a property \( \varphi \) that we should like to preserve under consequence, but not be able to say, for any given sentence, what exactly might *cause* the addition of that sentence to a set to fail to have the property. In a sense, negation’s structural role in classical logic is tied up with this sense of “causation” in that the negation of any sentence \( \alpha \) is that thing entailed by any set of sentences inconsistent with \( \alpha \). Without this entailment, we might say, adding \( \alpha \) to our set of premisses is entirely alright. Speaking more formally, the lack of non-trivial denials cuts off one possible route to finding the sorts of rules of inference that correspond to our theory of consequence. Without non-trivial, negation-like denial, for instance, we cannot have a negation-introduction type of rule, whereby any derivation of an absurdity from a set under addition of \( \alpha \) allows us to introduce a formula featuring the negation of \( \alpha \). As well, it can be difficult, formally, to demonstrate that a logic *does have* non-trivial or negation-like denial, even if it does. As such, we argue, it is necessary to move on from denial to another notion, which we will call *logical opposition*. As we shall see, this concept is free of these problems.

### 4.3 Moving Beyond Denial to Opposition

The key reason for the difficulties with the notion of denial, as we see it, is the concentration on *individual sentences*. In one respect, this is not surprising: most accounts of what negation (or any negation-like thing) must be involve such a focus, since we are used to thinking of negation as a unary connective, or at least as a function over single sentences. However, just as we have relaxed our notions of what a preservable property can be, we can relax our understanding of what “negation” can be. Thus, we define logical opposition in terms of *sets of sentences*; for convenience in doing so, we extend our \( \varphi \)-preserving consequence relation so that it can take sets on both sides, writing \( \Sigma \vdash_{\varphi} \Delta \) simply to
mean that $\forall \alpha \in \Delta$, $\Sigma \vdash \varphi \alpha$. (Again, the presumption here is always that $\varphi$ is Lindenbaum.)

**Definition 9** ($\varphi$-Opposed). For any sentence $\alpha$ and set $\Sigma$, $\Sigma$ is $\varphi$-opposed to $\alpha$, written $\text{Opp}(\alpha, \Sigma)$ iff:

1. **$\varphi$ Failure:** $\overline{\varphi}(\Sigma, \alpha)$.
2. **Minimality:** $\forall \Delta (\Sigma \vdash \varphi \Delta \Rightarrow [\Delta \vdash \varphi \Sigma \text{ OR } \varphi(\Delta, \alpha)])$.

That is, $\Sigma$ is $\varphi$-opposed to $\alpha$ just in case their union lacks the property $\varphi$, and $\Sigma$ is minimal with respect to $\varphi$-preserving consequence, in the sense that there is any set $\Delta$ that follows from $\Sigma$ and is also $\varphi$-incompatible with $\alpha$ is in fact simply equivalent to $\Sigma$. Given this definition, we now get a key result, corresponding to the property discussed above, whereby any set inconsistent with a sentence $\alpha$ entails its negation (Equation (1)).

**Theorem 2** (Opposite Entailment). For any sentence $\alpha$, sentence-set $\Sigma$, and Lindenbaum property $\varphi$, if $\overline{\varphi}(\Sigma \cup \{\alpha\})$, then there exists some set $\Delta$ such that $\text{Opp}(\alpha, \Delta)$ and $\Sigma \vdash \varphi \Delta$.

**Proof.** The proof is somewhat complex, and relies upon Zorn’s Lemma [Zorn, 1935]. Please see the Formal Appendix at the end of this paper.

Logical opposition also has one key advantage over logical denial, namely that that every sentence in a Lindenbaum context will have a $\varphi$-opposed set; furthermore, that set will be a contingent, non-trivial denial just in case $\alpha$ is itself a contingent, non-absurd sentence.

**Theorem 3** (Existence of Opposites). For any sentence $\alpha$, and Lindenbaum property $\varphi$, there exists some sentence-set $\Delta$ that is $\varphi$-opposed to $\alpha$, as follows:

1. If $\alpha$ is $\varphi$-tautological ($\forall \Sigma$, $\Sigma \vdash \varphi \alpha$), then $\Delta$ is any absurd set such that $\overline{\varphi}(\Delta)$.
2. If $\alpha$ is $\varphi$-absurd ($\forall \beta$, $\{\alpha\} \vdash \varphi \beta$), then $\Delta$ is any set of $\varphi$-tautologies ($\forall \Sigma$, $\Sigma \vdash \varphi \Delta$).
3. If $\alpha$ is neither $\varphi$-tautological, nor $\varphi$-absurd, but is $\varphi$-contingent, then $\Delta$ is some non-tautological, non-absurd set ($\varphi(\Delta)$ and $\exists \Sigma$, $\Sigma \not\vdash \varphi \Delta$).

**Proof.** Please see the Formal Appendix at the end of this paper.

Thus, while not every system will have denial, let alone non-trivial denial, every $\varphi$-preserving consequence relation will generate appropriate logical opposed sets for any sentence $\Delta$, so long as $\varphi$ is Lindenbaum.

Finally, we note one more important property of logical opposition as we have defined it. As we have discussed, a general preservationist system generally lacks bivalence, since it is easily possible that maximal-$\varphi$ sets $\Sigma^*$ exist such that $\{\alpha, \neg \alpha\} \subseteq \Sigma^*$. However, our new definition of $\varphi$-opposed sets allows us to recapture a somewhat weaker, but
still very useful, form of classical strong bivalence, in the sense that, for any maximal-$\varphi$ set $\Sigma^*$ and sentence $\alpha$, either $\alpha$ or some $\varphi$-opposed set, but not both, is contained in $\Sigma^*$.

**Theorem 4** (Weak Bivalence). For any maximal-$\varphi$ set $\Sigma^*$, and any sentence $\alpha$,

$$\alpha \in \Sigma^* \iff \forall \Delta \left[ Opp(\alpha, \Delta) \Rightarrow \Delta \not\subseteq \Sigma^* \right].$$

*Proof.* Please see the Formal Appendix at the end of this paper. \qed

### 5 Discussion and Conclusions

We have now shown three important things about logically opposed sets of sentences for Lindenbaum properties $\varphi$.

First, as Theorem 2 demonstrates, if some set of premisses is inconsistent with respect to $\varphi$ with a sentence $\alpha$, then that set entails some logically opposed set for $\alpha$. Just as in classical logic negation “causes” inconsistency, in that the negation of a sentence already follows from any set inconsistent with the sentence, the logically opposed sets in some sense “cause” the failure to have our new property $\varphi$. Secondly, by Theorem 3, every sentence has a logically opposed set, just as every sentence has a negation in classical logic. This distinguishes our idea from that of denial, which sometimes fails outright; in addition, we have that the opposed sets are in some sense the *right ones*, especially in that any contingent, non-absurd sentence has a similarly contingent and non-absurd opposed set. Lastly, Theorem 4 shows us that opposition restores a relevant form of bivalence to maximal-$\varphi$ sets. Each of these features corresponds to an important characteristic of negation as it is normally understood in logic.

Furthermore, it is easy to see that if we remain in a purely classical context, where the property $\varphi$ is classical consistency, the idea of an opposed set actually corresponds to the negation of a sentence (or at least the singleton set containing the negation). For any sentence $\alpha$, the set consisting of its negation, $\{\neg \alpha\}$ has precisely the properties given in Definition 9: the set $\{\alpha, \neg \alpha\}$ is inconsistent, and by the bivalence property discussed above (Eq. (1)), $\{\neg \alpha\}$ is minimal in that anything that follows from it is only inconsistent with $\alpha$ if it entails that negation. We thus argue that this idea captures something that is both of real importance in preservationist logical contexts, and broadens the notion of negation and its structural role appropriately.

Other attempts to broaden the definition of negation have generally retained the requirement that the negation-like object for any sentence $\alpha$ be another sentence [Gabbay and Sergot, 1986; Gabbay and Hunter, 1999; Payette, 2009]. In many ways, our results simply extend some of these ideas in new directions by moving from a single sentence to a set of sentences. We would argue that there is no reason to balk at doing so. As we have stressed, the logical opposite
does not replace or eliminate the usual sentential sense of negation; rather, it simply identifies something that plays the logical role of sentential negation, and even reduces to it in the classical limit. Particularly because logically opposite sets are guaranteed to exist, and will do much the work of negation in Lindenbaum contexts, we think they are worthy objects of study, whether we call them “negations” or not.

Finally, we note one fact that suggests some perhaps deeper truth. The paper begins with the Tukey-Teichmüller result that any compact property can be extended to maximality (Lemma 1). Toward the end, in proving our result about the entailment of logically opposed sets (Theorem 2), we employed Zorn’s Lemma. It is well known that these two Lemmas are equivalent to the Axiom of Choice, and to one another. This indicates that there is some important connections between properties that can be extended to maximality and the existence of a negation-like object in a logical context. While we do not care to speculate on this connection here, it is well worth pondering.
Formal Appendix: Proofs of Outstanding Claims

Lemma 1 (Tukey-Teichmüller): For any compact property \( \varphi \) and set \( \Sigma \) such that \( \varphi(\Sigma) \), \( \Sigma \) has some maximal-\( \varphi \) extension, \( \Sigma^* \).

Proof. Our proof relies on the presumption that our language \( L \) is denumerable, and thus that we can sensibly arrange its sentences in some fixed order \( \langle \alpha_0, \alpha_1, \ldots \rangle \). Given such an ordering, and any set \( \Sigma \subseteq L \) such that \( \varphi(\Sigma) \), we can constructively define an inductive sequence of sets, \( \langle \Sigma_0, \Sigma_1, \ldots \rangle \) as follows:

\[
\Sigma_0 := \Sigma \\
\Sigma_{n+1} := \begin{cases} 
\Sigma_n \cup \{ \alpha_n \} & \text{if } \varphi(\Sigma_n \cup \{ \alpha_n \}) \\
\Sigma_n & \text{else.}
\end{cases}
\]

We then define \( \Sigma^* := \bigcup \Sigma_i \). Obviously, \( \Sigma \subseteq \Sigma^* \); we need only show that the union \( \Sigma^* \) is maximal-\( \varphi \).

To show that \( \varphi(\Sigma^*) \) we consider an arbitrary finite subset \( \Sigma' \subseteq \Sigma^* \). By our construction, there exists some value \( n \) such that \( \Sigma' \subseteq \Sigma_n \); furthermore, each such \( \Sigma_n \) has property \( \varphi \) by definition. Thus, by the compactness of \( \varphi \), we know that \( \varphi(\Sigma') \), since it is a finite subset of \( \Sigma_n \). Therefore, since \( \Sigma' \) was arbitrary, all finite subsets of \( \Sigma^* \) have property \( \varphi \), and by compactness it follows that \( \varphi(\Sigma^*) \).

To show that \( \Sigma^* \) is maximal we consider any formula \( \beta \) such that \( \varphi(\Sigma^* \cup \{ \beta \}) \), and show that \( \beta \in \Sigma^* \). Any such formula is part of our language \( L \), and so there exists some value \( m \) such that \( \beta = \alpha_m \) in our enumeration of \( L \). Now consider the set \( \Sigma_m \) from our inductive definition. Since any finite subset of \( \Sigma_m \cup \{ \alpha_m \} \) is also a finite subset of \( \Sigma^* \cup \{ \alpha_m \} \), compactness of \( \varphi \) means that \( \varphi(\Sigma_m \cup \{ \alpha_m \}) \) as well. Thus, by definition, \( \Sigma_{m+1} = \Sigma_m \cup \{ \alpha_m \} \), and since \( \Sigma_{m+1} \subseteq \Sigma^* \), we have that \( \alpha_m = \beta \in \Sigma^* \), as required. \( \square \)

Proposition 5: For any \( \Sigma \) and \( \alpha, \Sigma \models_\varphi \alpha \) if and only if: \( \forall \Sigma'[(\Sigma' \supseteq \Sigma \& \varphi(\Sigma')) \Rightarrow \varphi(\Sigma', \alpha)] \).

Proof. For the rightward direction, assume that \( \Sigma \models_\varphi \alpha \) for our Lindenbaum property \( \varphi \). Now \( \Sigma \) either has property \( \varphi \) or it does not. If it does not, then there can be no superset \( \Sigma' \supseteq \Sigma \) such that \( \varphi(\Sigma') \). If there were such a \( \Sigma' \), then since \( \varphi \) is Lindenbaum, it would have some maximal-\( \varphi \) extension \( \Sigma^* \); however, any such \( \Sigma^* \) would also be a maximal-\( \varphi \) extension of \( \Sigma \), and again because \( \varphi \) is Lindenbaum, this would mean \( \varphi(\Sigma) \), contrary to hypothesis. Thus, if \( \varphi(\Sigma) \), the right-hand side of the if and only if is satisfied vacuously. Now, if \( \varphi(\Sigma) \) does hold, then consider any \( \Sigma' \supseteq \Sigma \) such that \( \varphi(\Sigma') \), and consider arbitrary maximal-\( \varphi \) extension \( \Sigma^* \in [\Sigma']^* \). By definition, \( \Sigma^* \) is also a maximal-\( \varphi \) extension of \( \Sigma \), and so by Definition 5, \( \alpha \in \Sigma^* \). Thus, since \( \Sigma^* \) was arbitrary, \( \alpha \) is a member of every maximal-\( \varphi \) extension of \( \Sigma^* \), and so \( \Sigma' \models_\varphi \alpha \), as required.

For the leftward direction, assume that \( \Sigma \not\models_\varphi \alpha \). Now by Proposition 4, \( \varphi(\Sigma) \), and so there must exist some maximal-\( \varphi \) extension \( \Sigma^* \supseteq \Sigma \) such that \( \alpha \notin \Sigma^* \). Since \( \varphi(\Sigma^*) \) holds by definition, and maximality entails that \( \varphi(\Sigma^*, \alpha) \), we have that \( \alpha \) fails to preserve \( \varphi \) for some \( \varphi \)-preserving extension of \( \Sigma \). \( \square \)

Proposition 6 (Structure of \( \models_\varphi \)): The consequence relation \( \models_\varphi \) has the following structural features (each given in terms of the consequence-set \( \text{Cn}_\varphi^n(\Sigma) \)):

1. Reflexivity: \( \forall \Sigma \ [\Sigma \subseteq \text{Cn}_\varphi^n(\Sigma)] \).
2. Monotonicity: \( \forall \Sigma, \Delta \ [\Sigma \subseteq \Delta \Rightarrow \text{Cn}_\varphi^n(\Sigma) \subseteq \text{Cn}_\varphi^n(\Delta)] \).
3. Idempotence: \( \forall \Sigma \ [\text{Cn}_\varphi^n(\text{Cn}_\varphi^n(\Sigma)) = \text{Cn}_\varphi^n(\Sigma)] \).
4. Transitivity: \( \forall \Sigma, \Delta \ [\Delta \subseteq \text{Cn}_\varphi^n(\Sigma) \Rightarrow \text{Cn}_\varphi^n(\Delta) \subseteq \text{Cn}_\varphi^n(\Sigma)] \).
Proof. We prove that $\models$ has each of the structural features in turn.

**Reflexivity:** Consider any set $\Sigma$, and arbitrary sentence $\alpha \in \Sigma$. For any maximal-$\varphi$ extension $\Sigma^* \in [\Sigma]^*_\varphi$, $\Sigma \subseteq \Sigma^*$ by definition, and thus $\alpha \in \Sigma^*$, for each such $\Sigma^*$. Thus, $\alpha \in Cn^\varphi_\Sigma(\Sigma)$ by definition, and since $\alpha$ was an arbitrary element of $\Sigma$, $\Sigma \subseteq Cn^\varphi_\Sigma(\Sigma)$.

**Monotonicity:** Consider any sets $\Sigma, \Delta$, where $\Sigma \subseteq \Delta$. Either $\varphi(\Delta)$ or not. If not, then by Proposition 4, $Cn^\varphi_\Sigma(\Delta) = L$, and $Cn^\varphi_\Sigma(\Sigma) \subseteq Cn^\varphi_\Sigma(\Delta)$ trivially. On the other hand, if $\varphi(\Delta)$, then there exists some maximal-$\varphi$ extension of $\Delta$, $\Delta^*$. For such any extension, $\Delta \subseteq \Delta^*$, and so $\Sigma \subseteq \Delta^*$, and $\Delta^*$ is a maximal-$\varphi$ extension of $\Sigma$. Thus, for any $\alpha \in Cn^\varphi_\Sigma(\Sigma), \alpha \in \Delta^*$ by deductive closure (Proposition 3), and so $\alpha \in Cn^\varphi_\Sigma(\Delta)$, and thus $Cn^\varphi_\Sigma(\Sigma) \subseteq Cn^\varphi_\Sigma(\Delta)$.

**Idempotence:** Consider any set $\Sigma$. Trivially, if $\varphi(\Sigma)$, then $Cn^\varphi_\Sigma(\Sigma) = L$ by Proposition 4. Furthermore, since $\varphi$ is a Lindenbaum property and so $\varphi(L)$ (otherwise, all sets $\Sigma$ would have property $\varphi$, and it would not be genuine); thus $Cn^\varphi_\Sigma(L) = L$ as well. Trivially, then, $Cn^\varphi_\Sigma(Cn^\varphi_\Sigma(\Sigma)) = Cn^\varphi_\Sigma(L) = L$.

Thus, we suppose that $\varphi(\Sigma)$. Since $\vdash$ is reflexive, we know that $\Sigma \subseteq Cn^\varphi_\Sigma(\Sigma)$. Therefore, since $\vdash$ is also monotonic, we have that $Cn^\varphi_\Sigma(\Sigma) \subseteq Cn^\varphi_\Sigma(Cn^\varphi_\Sigma(\Sigma))$. Now consider any $\beta \in Cn^\varphi_\Sigma(Cn^\varphi_\Sigma(\Sigma))$. By definition, $\beta$ is an element of any maximal-$\varphi$ extension $\Sigma^*$ of $Cn^\varphi_\Sigma(\Sigma)$. Let $\Sigma^*$ be a maximal-$\varphi$ extension of $\Sigma$. Then, again by definition, $Cn^\varphi_\Sigma(\Sigma)$ is a subset of any such $\Sigma^*$, and so $\Sigma^*$ is a maximal-$\varphi$ extension of $Cn^\varphi_\Sigma(\Sigma)$ as well. Thus $\beta \in \Sigma^*$ for any such $\Sigma^*$, which is to say that $\beta \in Cn^\varphi_\Sigma(\Sigma)$. Therefore, since $\beta$ was an arbitrary element, $Cn^\varphi_\Sigma(Cn^\varphi_\Sigma(\Sigma)) \subseteq Cn^\varphi_\Sigma(\Sigma)$, and so $Cn^\varphi_\Sigma(\Sigma) = Cn^\varphi_\Sigma(Cn^\varphi_\Sigma(\Sigma))$ as required.

**Transitivity:** Consider any sets $\Delta$ and $\Sigma$, where $\Delta \subseteq Cn^\varphi_\Sigma(\Sigma)$. Since $\models$ is monotonic, we know that $Cn^\varphi_\Sigma(\Delta) \subseteq Cn^\varphi_\Sigma(Cn^\varphi_\Sigma(\Delta))$. And so, since $\models$ is idempotent, we have that $Cn^\varphi_\Sigma(\Delta) \subseteq Cn^\varphi_\Sigma(\Sigma)$.

\[ \square \]

**Theorem 2 (Opposite Entailment):** For any sentence $\alpha$, sentence-set $\Sigma$, and Lindenbaum property $\varphi$, if $\varphi(\Sigma \cup \{ \alpha \})$, then there exists some set $\Delta$ such that $\text{Opp}(\alpha, \Delta)$ and $\models \alpha \varphi \Delta$.

**Proof.** We begin by noting that if $\alpha$ is a $\varphi$-tautology, then $\varphi(\Sigma \cup \{ \alpha \})$ only if $\varphi(\Sigma)$, and so the result follows easily. It is also easy to see that if $\alpha$ is $\varphi$-absurd, so $\varphi(\{ \alpha \})$, then any set of $\varphi$-tautologies is logically opposed to $\alpha$, and the result again follows trivially. We thus concentrate upon the case where $\alpha$ is a contingent, non-absurd sentence.

The proof relies upon Zorn’s Lemma [Zorn, 1935]. Briefly, let $X$ be any partially ordered set with ordering $\leq$. A chain in $X$ is any totally ordered subset $X' \subseteq X$ (so that any two items in the subset are directly comparable); a chain $X'$ has an upper bound just in case there exists some element $u' \in X$ such that $\forall x \in X', x' \leq u'$. Any element $m \in X$ is maximal just in case $\forall x \in X, x \leq m$. Zorn’s Lemma states that any partially ordered set in which every chain has an upper bound contains at least one maximal element. (A detailed proof can be found in [Halmos, 1974].)

For our set $\Sigma$ such that $\varphi(\Sigma \cup \{ \alpha \})$, define the following set of sets:

\[ \Sigma^+ = \{ [\Sigma']^*_\varphi : [\Sigma]^*_\varphi \subseteq [\Sigma']^*_\varphi \text{ and } (\forall \Sigma^* \in [\Sigma']^*_\varphi, \alpha \notin \Sigma^*) \}\]

That is, $\Sigma^+$ is the set of maximal-$\varphi$ extensions of any set $\Sigma'$ such that $\Sigma \models \varphi \Sigma'$ and $\varphi(\Sigma' \cup \{ \alpha \})$. Note that $\Sigma^+$ is partially ordered by subset inclusion ($\subseteq$). Now consider any totally ordered chain $\Delta \subseteq \Sigma^+$. It is straightforward that for any such $\Delta = \{ [\Sigma_1]^*_\varphi, [\Sigma_2]^*_\varphi, \ldots \}$, the union of its elements $\bigcup \{ [\Sigma_i]^*_\varphi \in \Delta \}$ is such that (a) $[\Sigma]^*_\varphi \subseteq \bigcup \{ [\Sigma_i]^*_\varphi \in \Delta \}$, and (b) $\forall \Sigma^* \in \bigcup \{ [\Sigma_i]^*_\varphi \in \Delta \}, \alpha \notin \Sigma^*$. Therefore, this union is also an element of the overall set: $\bigcup \{ [\Sigma_i]^*_\varphi \in \Delta \} \subseteq \Sigma^+$.

Thus, since this union is trivially an upper bound on the arbitrary chain $\Delta$, any such chain in $\Sigma^+$ has an upper bound, by Zorn's Lemma we have that there exists some maximal element $[\Sigma_m]^*_\varphi \in \Sigma^+$ such that $\forall [\Sigma']^*_\varphi \in \Sigma^+, [\Sigma_m]^*_\varphi \subseteq [\Sigma']^*_\varphi \Rightarrow [\Sigma_m]^*_\varphi = [\Sigma']^*_\varphi$.

1. $[\Sigma]^*_\varphi \subseteq [\Sigma_m]^*_\varphi$, so $\Sigma \models \varphi \Sigma_m$.
2. $\forall \Sigma^* \in [\Sigma_m]^*_\varphi, \alpha \notin \Sigma^*$, so $\varphi(\Sigma_m \cup \{ \alpha \})$.
3. $\forall \Sigma' \exists m [\Sigma_m \models \varphi \Sigma' \text{ and } \varphi(\Sigma' \cup \{ \alpha \})] \Rightarrow \Sigma' \models \varphi \Sigma_m$.  

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That is, $\Sigma \models_{\varphi} \Sigma_m$ and $Opp(\alpha, \Sigma_m)$ as required.

**Theorem 3 (Existence of Opposites):** For any sentence $\alpha$, and Lindenbaum property $\varphi$, there exists some sentence-set $\Delta$ that is $\varphi$-opposed to $\alpha$, as follows:

1. If $\alpha$ is $\varphi$-tautological ($\forall \Sigma, \Sigma \models_{\varphi} \alpha$), then $\Delta$ is any absurd set such that $\varphi(\Delta)$.
2. If $\alpha$ is $\varphi$-absurd ($\forall \beta, \{\alpha\} \models_{\varphi} \beta$), then $\Delta$ is any set of $\varphi$-tautologies ($\forall \Sigma, \Sigma \models_{\varphi} \Delta$).
3. If $\alpha$ is neither $\varphi$-tautological, nor $\varphi$-absurd, but is $\varphi$-contingent, then $\Delta$ is some non-tautological, non-absurd set ($\varphi(\Delta)$ and $\exists \Sigma, \Sigma \not\models_{\varphi} \Delta$).

**Proof.** For the first case, if $\alpha$ is $\varphi$-tautological, then $\forall \Delta, \varphi(\Delta \cup \{\alpha\}) \iff \varphi(\Delta)$. So, consider any $\Delta'$ such that $\Delta \vdash_{\varphi} \Delta'$; again such a set will be $\varphi$-inconsistent with $\alpha$ just in case it is itself absurd $\varphi(\Delta')$, in which case $\Delta' \vdash_{\varphi} \Delta$ and $Opp(\alpha, \Delta)$ as required.

For the second case, if $\alpha$ is $\varphi$-absurd, then it has no maximal-$\varphi$ extensions, and so $\varphi(\Delta \cup \{\alpha\})$, for any set $\Delta$ whatsoever. Consider any set of $\varphi$-tautologies, $\Delta$. Since for any set $\Sigma, \Sigma \models_{\varphi} \Delta$, the minimality clause of Definition 9 is vacuously satisfied, and so $Opp(\alpha, \Delta)$ as required.

For the third case, suppose that $\alpha$ is contingent but not $\varphi$-absurd, so that there exists some set $\Sigma$ such that $\Sigma \not\models_{\varphi} \alpha$. Now, there must exist some set $\Delta$ that is not wholly tautological such that $\varphi(\Delta)$, but $\varphi(\Delta \cup \{\alpha\})$. For suppose not: then $\alpha$ is $\varphi$-consistent with all non-absurd sets (it must by definition be consistent with wholly $\varphi$-tautological sets), and so $\alpha \in \Sigma^*$ for all maximal-$\varphi$ sets $\Sigma^*$, and $\alpha$ is a $\varphi$-tautology, contrary to hypothesis. So, since such a set $\Delta$ exists, by Theorem 2, there exists some set $\Delta'$ such that $\Delta \vdash_{\varphi} \Delta'$ and $Opp(\alpha, \Delta')$; furthermore, this set $\Delta'$ must be non-absurd (since it follows from a non-absurd set) and non-tautological by the same reasoning as above.

**Theorem 4 (Weak Bivalence):** For any maximal-$\varphi$ set $\Sigma^*$, and any sentence $\alpha$,

$$\alpha \in \Sigma^* \iff \forall \Delta [Opp(\alpha, \Delta) \Rightarrow \Delta \not\subseteq \Sigma^*].$$

**Proof.** Consider arbitrary sentence $\alpha$ and maximal-$\varphi$ set $\Sigma^*$, and suppose $\alpha \in \Sigma^*$. Now consider any set $\Delta$ such that $Opp(\alpha, \Delta)$, and suppose $\Delta \subseteq \Sigma^*$. In that case $(\Delta \cup \{\alpha\}) \subseteq \Sigma^*$, and since $\varphi(\Delta \cup \{\alpha\})$, it follows that $\varphi(\Sigma^*)$, which is absurd. Thus, since $\Delta$ was arbitrary, $\forall \Delta [Opp(\alpha, \Delta) \Rightarrow \Delta \not\subseteq \Sigma^*].$

Now suppose that $\alpha \not\in \Sigma^*$. Then by definition of maximality, $\varphi(\Sigma^* \cup \{\alpha\})$, and by Theorem 2, there exists some $\Delta$ such that $Opp(\alpha, \Delta)$ and $\Sigma^* \models_{\varphi} \Delta$. So, by deductive closure, $\Delta \subseteq \Sigma^*$, as required.
References


