1 Introduction

Why you'll hate this class

• It's so tedious!
• I could do this in Java!
• This is so weird!
What this class gives you

• The vocabulary to discuss languages
• Experience now with what may come later
  – Java is a fine teaching language
  – And it’s comfortable for industry uses
  – But remember - it was once the cutting-edge technology
• What will be in the next five programming languages?
  – Career-focused, not first-job-focused

What we’ll do

• Name and compare the ideas behind different languages
• Experience programming languages different to those you’ll see elsewhere in the CS curriculum
  – Functional programming in Haskell
  – Object-oriented programming in Scala
  – Logic programming in Prolog
  – And we will see examples in other languages including Java, Common Lisp and Perl

Assessed work in CS421

• About 8 quizzes
• On-paper homework
  – Bring it to class; sometimes I’ll ask you to turn it in, to be counted as a small quiz
• Programming homework and projects
  – Probably two major projects with one in Haskell, one in Scala
• A final exam

Assessed work in CS521

• Programming projects
  – Probably two with one in Haskell, one in Scala
• Additional reading and assignments on material beyond the undergrad level
• A final exam
Obligatory administration

- There’s a syllabus - read it!
  - There’s a D2L quiz about the syllabus due Monday
  - Note in particular: absences for university travel (inc. sports) due this week if it is to be excused

- There’s a course website — [cs.uwlax.edu/~jmaraist/420-spring-17](cs.uwlax.edu/~jmaraist/420-spring-17)
  - Check it frequently for news, announcements, assignments, schedule, notes etc.
  - D2L for some assignment submission, some quizzes - but not announcements
  - There’s an RSS feed attached to the web site

- There’s a study guide — linked from the web site
  - Contains lecture slides and exercises
  - First several sections available now, others will be announced via course website

- There’s email: jmaraist@uwlax.edu
  - Check it frequently for feedback on assignments, Q&A
  - Expect replies within a (business) day (but typically faster)
  - Administrative stuff always by email

- There are open-door hours
  - On the first slides, on the web page
  - Or by appointment, but email at least a (business) day ahead
    * Always include the questions you’ll want to discuss: So I can be prepared, because advising and paperwork often require no meeting, because every meeting needs an agenda
  - But I am not able to linger after class for extra questions, since I have a class immediately after, usually in another building

- Always silence your gadgets
  - Consider an app to do it for you so you don’t forget

- When you pick your seat, please:
  - Computers and handhelds to the back
  - Latecomers and early-leavers to the aisle

Class materials

The textbook is *Programming Language Pragmatics*, Michael L. Scott

- Get yourself a copy of the book
- Undergraduates: use the textbook rental service
- Graduates:
  - The bookstore will sometimes have used copies; ask at the back desk
  - You can often find cheap copies on Amazon or other online stores
  - In the past, grad students who tried to do without the book (and with an old edition) have complained about the difficulties in getting work done

See the course homepage for information about other resources

- Books on reserve in the library
- Online tutorial sites
- Other references
On class scheduling

- The required 400-level classes — 421, 441 and 442 — are all difficult and time-intensive classes
  - It can be a challenge to manage two of them at once
  - It is rarely a good idea to take all three at once

2 Specifying syntax

2.1 Regular expressions

What is there to a language?

Syntax

- The form of a program
- Essentially two aspects of syntax:
  - How you spell stuff — specified by a regular expression (regex)
    * Basic strings
    * Concatenation of two or more regexes
    * Choice from alternative regexes
    * Arbitrarily many repetitions of some regex
  - How you put correctly-spelled stuff together — specified by a context-free grammar (CFG), often in Backus-Naur form (BNF)
    * Give a starting symbol, other nonterminal symbols which are not part of the language
    * Rules say how a nonterminal may be rewritten to a string of other nonterminals and terminals

Semantics

- The meaning of a program
  - Most of this class focuses on language semantics

Writing down regular expressions

A language is a just a set of strings

- It can be finite (the first names of the people in this class) or infinite (phrases used to represent natural numbers)
- Any plain character in the language we’re generating is a regular expression by itself

Regular expressions are a notation for writing languages

- The empty string is a regular expression. Write it this way: $\varepsilon$
- Write two regular expressions next to each other to represent concatenation
- Separate alternatives with a vertical bar
- Use the Kleene star as a suffix for repetitions
- Use parentheses to make grouping clear

Followup reading: Scott, Ch. 1
Exercise 2.1. Write regular expressions for the following languages:

1. Strings which consist of an even number of "r"s
2. Strings which start with a lower-case letter, and are followed by any alphanumeric characters
3. Strings consisting of a number of even-valued digits with a single "E" before all of them
4. Strings consisting of one or more odd digits with a single "o" in front of them

Exercise 2.2. Write regular expressions over the alphabet \{0, 1\} for the following languages [Sipser]:

1. Strings which begin with a 1 and end with a 0
2. Strings which contain at least three 1’s (not necessarily in order)
3. Strings which contain the substring 0101
4. Strings which are at least three characters long, and have 0 as their third character
5. Strings which start with 0 and have odd length, or start with 1 and have even length
6. Strings which do not contain the substring 110
7. Strings which are at least five characters long
8. Any string except 11 or 111
9. Strings where every odd position (starting counting from 1) is a 1
10. Strings which contain at least two 0’s and at most one 1
11. Either the empty string or 0
12. Strings which contain an even number of 0’s, or exactly two 1’s
13. All strings except the empty string

Exercise 2.3. Scott, Exercise 2.1

Exercise 2.4. Write regular expressions for these languages:

1. All strings over \{0, 1, 2\} except for 2 and 10
2. All sequences of lower-case letters except for three strings: file, for and from [Scott, Exercise 2.3]

Exercise 2.5. Describe in English the language generated by the regular expression $a^*(ba^*ba)^*$. Your description should be high-level — the simple intuition about the strings, rather than a transliteration of the expression into English. [Scott, Ex. 2.9(a)]

2.2 Finite automata

Regular expressions generate, automata recognize

A finite automaton is a simple, idealized machine which corresponds to a language

- It has a number of states
  - One is initial
  - One is final
• When there is an item of input, the machine transitions from one state to another
  – Each transition is based on a single input item — no peeking ahead!
  – The number of states, transitions and transition labels must be finite

• If a string’s characters give transitions from the initial state to a final state, then the automaton accepts the string as part of its language
  – Otherwise, it rejects the string

Depicting regular expressions
We usually draw an automaton graphically

• States are circles
  – The initial state is marked with an arrow pointing to it
  – The final states are double-circled

• Transitions are arrows from one state to another
  – Labelled with its character
  – An arrow can start and end at the same state
  – To avoid the clutter of multiple arrows, can draw one arrow with multiple labels

Exercise 2.6. Which of these strings does the automata below accept: a, b, c, ab, bb, ba, cb, cba, cab?

Exercise 2.7. Write finite automata (using the circles-and-arrows notation) for each of the languages in Exercise 2.2.

Deterministic or nondeterministic?
A finite automaton is deterministic if for every state and input symbol, there is at most one possible transition

• Otherwise, the automaton is nondeterministic

• A nondeterministic automaton accepts a string if any series of transitions from initial to final state exists

• With nondeterministic automata, it is acceptable to label transitions with the empty string, or with multi-character strings
• It is *always* possible to write a deterministic finite automaton which corresponds to a nondeterministic automaton
  – But the nondeterminist automaton might be more concise
• It is *always* possible to write a finite automaton for the language of a regular expression
• But it is *not* possible to find a finite automaton for *every* language

**Followup reading:** Scott, Sec. 2.1-2.2

**Exercise 2.8.** Scott, Exercise 2.4

**Exercise 2.9.** Make sure each of the automata in the Exercise 2.7 are deterministic

**Exercise 2.10.** Scott, Exercise 2.2

### 2.3 Grammars and parsing

#### 2.3.1 Context-free grammars

**From regular expressions to grammars**

Regular expressions are one way define a language

• *Context-free grammars* written in the Backus-Naur form (BNF) are another

• Grammars generate a language based on rules for *rewriting* special symbols which are not in the language’s alphabet into other strings
  – The rules should eventually let us rewrite to a string which uses *only* characters in the language’s alphabet
  – The special symbols are called *nonterminals*, and the characters in the language’s alphabet are called *terminals*

**Writing down grammars**

• There’s a starting nonterminal symbol, with a rule for the form it can have:
  – S → hello goodbye

• There may be other nonterminals, with rules that refer to each other
  – S → T goodbye
    T → hello

• Use a vertical bar to separate alternative choices, or give multiple rules for a nonterminal
  – S → T goodbye
    T → bonjour | guessgott | hola

• Extended BNF (EBNF) includes the Kleene star and plus notations
Exercise 2.11. Consider this grammar $G$, with start symbol $R$ [Sipser]:

\[
G = 
\begin{align*}
R & \rightarrow XRX \mid S \\
S & \rightarrow aTb \mid bTa \\
T & \rightarrow XTX \mid X \mid \varepsilon \\
X & \rightarrow a \mid b
\end{align*}
\]

1. Give three examples of strings in $L(G)$
2. Give three examples of strings not in $L(G)$
3. True or false: can $T$ rewrite to $T$?
4. True or false: can $T$ rewrite to $aba$?
5. True or false: can $T$ rewrite to $abb$?
6. True or false: can $T$ rewrite to $ababa$?
7. True or false: can $R$ rewrite to $ababa$?
8. True or false: can $X$ rewrite to $XX$?
9. Describe $L(G)$ in English

Exercise 2.12. Give context-free grammars that generate the following languages over the alphabet $\{0, 1\}$ [Sipser]

1. Strings which begin with a $1$ and end with a $0$
2. Strings which contain at least three $1$’s (not necessarily in order)
3. Strings which contain the substring $0101$
4. Strings which start and end with the same symbol
5. Strings whose length is odd
6. Strings whose length is odd and whose middle symbol is $0$
7. Strings which contain the same number of $1$’s as $0$’s
8. Strings which contain more $1$’s than $0$’s
9. Strings which are palindromes

Exercise 2.13. Write an unambiguous context-free grammar that generates exactly the same language as the regular expression $a^*(ba^*ba^*)^*$. [Scott, Ex. 2.9(b)]

Exercise 2.14. Describing a grammar’s language in plain English: Scott, Exercise 2.12(a), 2.15(a)

Regex vs. grammars

- Every language that can be written as a regex can be written as a CFG
- What about the reverse?
- CFGs give a sort of simple memory that a regex does not have
• The same-number-as and palindrome examples cannot be written as a regex
• Although grammars are expressive enough for programming language syntax, there are nonetheless languages which they cannot express...
  – Cliffhanger! To be resolved in CS453/553

**Exercise 2.15.** Rewrite your regular expressions from Exercise 2.2 as context-free grammars.

**Parse trees**
To demonstrate that a string really is generated by a grammar, we produce a *parse tree*
• Each internal node labelled with a nonterminal
  – Starting symbol at the root
• Each leaf labelled with a terminal
• If there is a rule \( M \rightarrow u_1 u_2 \ldots u_n \), then a node labelled \( M \) could have \( n \) children labelled \( u_1 \) through \( u_n \)

**Followup reading:** Scott, Sec. 2.3 intro (to start of Sec. 2.3.1)

**Exercise 2.16.** Using the grammar of Exercise 2.11 give parse trees for these strings: \( babb \), \( babbb \), \( aababb \).

**Exercise 2.17.** Scott, Exercise 2.12(b)

**Exercise 2.18.** Scott, Exercise 2.13(a)

**Exercise 2.19.** Scott, Exercise 2.15(b)

2.3.2 Grammar properties

**Some properties of operators**

**Properties**
• Fixity: infix, prefix, postfix
• Arity
• Associativity
• Precedence

**Examples**
• In Java and C, `++` and `–` can be prefix or postfix
• Negation `–` is a prefix operator in most languages
• The arithmetic operators are usually infix
• Negation is *unary*, arithmetic operators are *binary*
  – The \(_?\sim : \_\) operator in C is *tertiary*
• In the standard interpretation of arithmetic expressions, addition, subtraction, etc. are *left-associative*
• In the standard interpretation of arithmetic expressions, multiplication *binds more tightly* than addition
Bad grammar
(Parentheses are literal, bars are metasyntactic)

\[
\text{Expr} \rightarrow \text{Expr} \> \text{Expr} | \text{Expr} \* \text{Expr} | \text{Expr} / \text{Expr} \\
| \text{Expr} + \text{Expr} | \text{Expr} - \text{Expr} | - \text{Expr} \\
| ( \text{Expr} ) | 0 | 1 | \ldots
\]

- What’s so bad about this grammar?
- How do we parse \(3+4*5\)?
  - Two ways: it is ambiguous
  - A grammar is ambiguous if it lets us build more than one parse tree for the same string

Exercise 2.20. Review the grammars you wrote in previous exercises. Which are ambiguous?

Better grammar

\[
\begin{align*}
\text{Expr} & \rightarrow \text{Expr} + \text{Product} | \text{Expr} - \text{Product} | \text{Product} \\
\text{Product} & \rightarrow \text{Product} \* \text{Power} | \text{Product} / \text{Power} | \text{Power} \\
\text{Power} & \rightarrow \text{Power} ^ \text{Basic} | \text{Basic} \\
\text{Basic} & \rightarrow ( \text{Expr} ) | - \text{Basic} | 0 | 1 | \ldots
\end{align*}
\]

- Is it still ambiguous for \(3+4*5\)?
- The additional structure constrains the possible derivations so that they are unique

2.3.3 Top-down parsing

Parsing
Grammars \textit{generate}, parsers \textit{recognize}

- Top-down or bottom-up?
- Top-down
  - Conceptually simple
  - More restrictions on the form of grammars which are allowed
  - Efficient
  - Can be implemented directly
- Bottom-up
  - Start with the terminal symbols, \textit{reduce} them into nonterminals
  - \(3+4*5\)
  - Lookahead
  - Usually implemented indirectly, using a generator, with a pushdown automation details via tables
- Lots of work has been done (and continues) on parsing — to come in CS442/542
Writing a top-down parser

Top-down parsers can be easy to write

- Each rule becomes a separate subroutine
- Each rule’s routine expects a string matching that rule body
  - Match terminals by finding them in the input
  - Match nonterminals by calling the corresponding subroutine

The difficulties:

- **Choice!** When there is a vertical bar |, or multiple rules for the same nonterminal, how does our program know which to pursue?
- **Left-recursion!** When a nonterminal expands to another of itself in the left-hand position

\[
\begin{align*}
\text{Expr} & \rightarrow \text{Expr} + \text{Product} \mid \text{Expr} - \text{Product} \mid \text{Product} \\
\text{Product} & \rightarrow \text{Product} \ast \text{Power} \mid \text{Product} / \text{Power} \mid \text{Power} \\
\text{Power} & \rightarrow \text{Power} ^ \text{Basic} \mid \text{Basic} \\
\text{Basic} & \rightarrow ( \text{Expr} ) \mid - \text{Basic} \mid 0 \mid 1 \mid ...
\end{align*}
\]

Removing left-recursion

So a lack of ambiguity is

- **Necessary** for a sensible grammar for a programming language
- But not yet **sufficient**

Must restructure the grammar to get rid of the left-recursion

- The Kleene star/plus operators of EBNF are often key tools
- We look ahead into the input to resolve choice
  - For efficiency, a solution should look only a single unit of input ahead before making each decision!

Followup reading: Scott, Sec. 2.3.1-2.3.2

Exercise 2.21. Rewrite the arithmetic grammar to remove left-recursion, and write a simple parser to evaluate strings representing arithmetic expressions.

\[
\begin{align*}
\text{Expr} & \rightarrow \text{Expr} + \text{Product} \mid \text{Expr} - \text{Product} \mid \text{Product} \\
\text{Product} & \rightarrow \text{Product} \ast \text{Power} \mid \text{Product} / \text{Power} \mid \text{Power} \\
\text{Power} & \rightarrow \text{Power} ^ \text{Basic} \mid \text{Basic} \\
\text{Basic} & \rightarrow ( \text{Expr} ) \mid - \text{Basic} \mid 0 \mid 1 \mid ...
\end{align*}
\]

3 Names and bindings

3.1 Scope

3.1.1 Stack model of execution

The stack model of execution

- The standard, basic organization of memory includes a **stack** and a **heap**
– The stack grows from one end of memory
– The heap grows from the other end of memory
  * (For now we’re thinking only about the stack, and will discuss the heap later)

• Each call to a subroutine pushes a frame onto a system stack.

• Each frame contains:
  – Storage for local variables
  – Storage for arguments
  – Pointer to top of previous frame

• The frame pointer is a CPU register used to point to the current frame

• This idealized version of the system stack organization gives us a form of operational semantics
  – Explain how we resolve variable references, parameter passing
  – Better than an English description, it’s a formal model

Example
For a program

```plaintext
sub f() {
  var z=2
  g(1)
}
sub g(x) {
  var y=3
  ...
}
```

When `f` calls `g`:

```
Frame for g

 Prev FP
 z 2

 Frame for f

 Top of stack
 y 3
 x 1

 Prev FP
```

Exercise 3.1.  Scott, Exercise 3.4, including Java examples

Exercise 3.2.  Scott, Exercise 3.9

How do nonlocal variables work in this model?

```plaintext
sub wrapper(x, y) {
  local z = somefn(x,y);
  nested sub inner(w, acc) {
    if (w<1) {
```

```plaintext
  }
  }
```
return fn2(z, acc);
} else {
    return inner(w-1, fn3(acc));
}

return inner(x, y);

How do we resolve inner's reference to z?

What about when the else branch recurs on inner?

Need an additional entry in the frame for the static pointer

• Points to the frame of the environment which encloses this frame in the source code
Followup reading: Scott, Sec. 3.1-3.2

Exercise 3.3. Scott, Exercise 3.6, in particular 3.6(b)

Exercise 3.4. Scott, Exercise 3.11: assume the P calls Q, and Q calls R.

3.1.2 Static and dynamic scope

What does this program print?

```plaintext
global z = 100;

sub f() {
    print z;
}

sub g(y) {
    val z = y;
    f();
}

main:
    g(10);
    print z;
```

- If these subroutines act like Java `static` methods?
- Or if they follow the static pointer as we discussed last time?
  - Then: 100
- But this is just one way of doing things!
  - A particular language could define the `scope` of name-binding differently

Finding z under a `static` scope rule

`Static` scope says that we should use the most `closely enclosing` binding to a name when accessing that name

```plaintext
global z = 100;

sub f() {
    print z;
}

sub g(y) {
    val z = y;
    f();
}

main:
    g(10);
    print z;
```

- Know (at compile time) that the in-scope reference for z from f is `one` enclosing scope outward
- So the code generated for f should refer through the static enclosure `once` to find the frame with z’s storage

14
• Print 100 both times

Finding z under a dynamic scope rule

*Dynamic scope says that we should use the most recent binding to a name when accessing that name*

• Conceptually, this means we should *follow the previous frame* until we find a frame which stores a value for that name

```plaintext
global z = 100;

sub f() {
    print z;
}

sub g(y) {
    val z = y;
    f();
}

main:
g(10);
print z;
```

• Not using the static enclosing-environment pointers

• The most recent binding to z is by g

• But this binding will end when g exits
  
  – So print 10 then 100
**Dynamic scope without search**

Implementations of *dynamic* scope avoid searching the stack by using frames to store hidden, out-of-scope bindings.

Then $f$ can read the current (dynamic) value of $z$ from the global frame.

**Followup reading:** Scott, Sec. 3.3

- **Exercise 3.5.** Scott, Exercise 3.5
- **Exercise 3.6.** Scott, Exercise 3.14
- **Exercise 3.7.** Scott, Exercise 3.18
- **Exercise 3.8.** Scott, Exercise 3.19

### 3.2 Parameter-passing

#### 3.2.1 Call-by-value

**Some vocabulary about parameters**

```python
def function1(x, y) = {
    return 2*x + 3*y;
}

// ...
val z = 10;
print function1(3, z);
```

- $x$ and $y$ are *formal* parameters
  - When considering `function1` by itself, we can make no assumptions about the values of $x$ and $y$
- $3$ and $z$ are *actual* parameters
  - When we call `function1`, they certainly do have specific values
- What is the relationship between formal and actual parameters?
  - That is, how does a language define that the former should be bound to the latter?

**Parameter-passing mechanisms**

You may never have considered the matter up for debate

- Java and C seem to have essentially the same behavior for their parameter-passing
- But just like static vs. dynamic scope, the choice of parameter-passing mechanism is a choice made by a language’s designers
Call-by-value

C’s parameter-passing mechanism is named *call-by-value*

- First evaluate the actual parameter (if it is an expression), and then pass that value.
- This is what we assumed in our lecture example for scope.
- Probably the most common, and in many ways the simplest, of the parameter-passing mechanisms we will see.

3.2.2 Call-by-reference

Call-by-reference

The traditional alternative to call-by-value in imperative languages

- Rather than the value itself being stored in the new subroutine’s frame, a *reference* to the location of that value is communicated
- Crucially, assignment to the formal parameter also update the actual parameter, since there is only a single stored value

For example, running `main` in

```python
def f(x) = {
    x=10
}
def main = {
    val b=5
    f(b)
}
gives
```

- Today, most commonly seen in C++
- In most languages with call-by-reference, the actual parameter *must* be a storage location
  - Not (for example) an arithmetic expression
- *Orthogonal* to many other choices, such as static vs. dynamic scope

Call-by-value and call-by-reference

- Given

```python
sub f(int x) {
    print x;
    x=3;
    return;
}
```
• What could happen when we evaluate

```java
int y=10;
f(y);
print y;
```

### 3.2.3 Call-by-sharing

**How does Java pass parameters?**

Scalar types are clearly passed by value, but what about object types?

• In a way, they are passed by value

```java
public void f(Object x) {
    x = new Object();
    // ...
}
```

The assignment does not change a caller’s variable

```java
final Object obj = "Hello";
f(obj);
println(obj); // Still shows Hello
```

• But in a way, they are passed by reference

```java
public void g(MyObj x) {
    x.setVal(x, 1.34);
    // ...
}
```

The assignment does change a caller’s field

```java
final MyObj obj = new MyObj(2.56);
println(obj.getVal()); // Shows 2.56
f(obj);
println(obj.getVal()); // Now shows 1.34
```

**Call-by-sharing**

We know enough about pointers to realize that what we are passing is a *pointer* to the actual object.

• And moreover that pointers are passed by value

• But the behavior is distinct enough from previous languages that we categorize Java’s mechanism as distinct from call-by-value

  – Named *call-by-sharing*
  
  – For non-simple types, pass a reference to some shared object
  
  – Side-effects altering the object are shared
  
  – But assignments to the formal parameter do *not* alter the actual parameter in calling routine
3.2.4 Call-by-copy-in/copy-out

Call-by-copy-in/copy-out
Like call-by-reference, concerns storage locations

• Before starting subroutine, evaluate the actual parameter
• Use the result value when starting subroutine
• When finishing subroutine, copy the final value of the formal parameter back to the actual parameter.

Exercise 3.9. Trace the evaluation of this main routine under both call-by-reference and call-by-copy-in/copy-out parameter-passing semantics.

```plaintext
int y=10;

sub g() {
    print y;
}

sub f(x) {
    x=3;
    g();
}

sub main() {
    f(y);
}
```

3.2.5 Call-by-name

Call-by-name
The parameter-passing mechanisms so far all start the same way

• "First, evaluate the expression given as the actual parameter"

But as usual, a language designer can choose differently.

Under call-by-name, formal parameters are substituted with the unevaluated actual parameter expression when a subroutine is called.

• So the expression may be evaluated multiple times
• But not until we reach each instance of the formal parameter
• And if the expression has side-effects, the effects may occur multiple times!

Call-by-name probably seems like the oddest of the mechanisms we’ve seen so far

• But it’s not a new idea — it was introduced into programming languages in the late 1950s with ALGOL60
• We’ll see an example from Scala shortly
• (And we’re not finished with parameter-passing mechanisms yet)
Scala example: Complaints!

```scala
object ComplaintCount {
  var num:Int = 0
  def another() = {
    num = num + 1
    num
  }
}

class Complaint {
  println("This is Complaint \#" + ComplaintCount.another())
  def sendTo(who:String) =
    println("Hey " + who + ", I have a complaint!")
}

An unsurprising example of complaining

```scala
object ComplaintCount {
  var num:Int = 0
  def another() = {
    num = num + 1
    num
  }
}

class Complaint {
  println("This is Complaint \#" + ComplaintCount.another())
  def sendTo(who:String) =
    println("Hey " + who + ", I have a complaint!")
}

object SenderBV extends App {
  tellAll(new Complaint())
  def tellAll(c:Complaint) {
    c.sendTo("Tom")
    c.sendTo("Dick")
    c.sendTo("Harry")
  }
}

• Not surprising when we run it: create a complaint, and send it around

  > scala SenderBV
  This is Complaint \#1
  Hey Tom, I have a complaint!
  Hey Dick, I have a complaint!
  Hey Harry, I have a complaint!
```
Call-by-name complaining

```scala
object ComplaintCount {
  var num:Int = 0
  def another() = {
    num = num + 1
    num
  }
}

class Complaint {
  println("This is Complaint "+ ComplaintCount.another())
  def sendTo(who:String) =
    println("Hey "+ who + ", I have a complaint!")
}

object SenderBN extends App {
  tellAll(new Complaint())
  def tellAll(c: => Complaint) {
    c.sendTo("Tom")
    c.sendTo("Dick")
    c.sendTo("Harry")
  }
}
```

• Writing `=>` as a prefix to a method parameter type means that the argument should be passed call-by-name
  
  – Not evaluated when the method is called
  – Evaluated fresh each time the method is used

• Now when we run it, we create a complaint each time we reference `c`

```scala
> scala SenderBN
This is Complaint #1
Hey Tom, I have a complaint!
This is Complaint #2
Hey Dick, I have a complaint!
This is Complaint #3
Hey Harry, I have a complaint!
```

Call-by-name can boil down boilerplate

If you have used Java’s `HashMap` classes before, you have probably written code like this:

```scala
V result;
if (map.containsKey(k)) {
  result = map.get(k);
} else {
  result = EXPR;
  map.put(k, result);
}
```

Scala’s equivalent to `HashMap` includes an extra method where the second parameter is call-by-name (indicated by the `=>`):
def getOrElse(key:K, default: => V): V
def getOrElseUpdate(key:K, defaultValue: => V): V

Call-by-name allows these common patterns to be more directly supported in the language

**Call-by-name without side effect**

What would call-by-name mean in the context of Haskell?

- Remember that Haskell does not have side-effects
- Does this insight let us optimize call-by-name?
- We could:
  1. Wait until a formal parameter is used before we evaluate it
  2. Share the result of the first evaluation among the other duplications of the actual parameter
- This strategy is known as *call-by-need,* or *lazy evaluation*
  - In fact, Haskell is defined to be a lazy language
  - We will see how:
    * Haskell associates laziness with data type constructors as well as with function application
    * Laziness allows much greater expressiveness when programming

3.2.6 Lecture 35 — Macros

**Macros**

- Not all applications of functions to arguments must take place at runtime
- A "function" that generates new source text from arguments is called a *macro*
- Macro facilities are fairly common, but there is great variability in what they can do
  - On one end, the C preprocessor performs simple text substitution
  - At the other end, Common Lisp allows arbitrary Lisp code to be executed at compile time to calculate source code
  - Haskell and Scala also recently added macro systems, which we might try out.
    * Which is at odds with the book’s claim that macros are anachronistic.

**C macros**

Just simple text substitution

```c
#define LINE_LEN 80
#define PI 3.141592651358979323846264338327950L
#define DIVIDES(a,n) (!((n) % (a)))
#define SWAP(a,b) {int tmp = (a); (a) = (b); (b) = tmp; }
#define MAX(x,y) ((x)<(y) ? (y) : (x))
```

- Was very useful for global or program constants
- Avoids overhead of function calls
- Note the extra parentheses
- What if \(a\) or \(b\) contain a reference to \(t\) from some surrounding scope?
- What if we call \(\text{MAX}(++m, ++n)\)?
  - Rewrites to \(((++m)<(++n) ? (++) : (++)\)
  - Would it be a surprise when one variable is incremented twice?
Some things to know about Lisp

- Lisp uses prefix notation: all operators are written with the function first:

  (+ 3 x (* 5 y))
  (append (list 1 2 x) y (list z 8 9))

- The parentheses are for invocation, not grouping
  - Not optional
  - Extras not allowed
  - If you play with Lisp, make your editor highlight matching parentheses

- Lisp has a defconstant form, so we wouldn’t use its macros for LINE_LEN or PI.

Lisp macros

(Defmacro divides (a n)
 '(zerop (mod ,n ,a)))

- The backtick ‘quotes a piece of syntax to be inserted by the compiler.

- The comma , injects syntax within the quoted expression.

Avoiding name capture

(Defmacro swap (x y)
 (let ((tmp (gensym)))
   '(let ((,tmp ,x))
     (setf ,x ,y
          ,y ,tmp))))
(Defmacro max (x y)
 (let ((xval (gensym))
        (yval (gensym)))
   '(let ((,xval ,x)
          ,yval ,y)
     (if (< ,xval ,yval)
         ,yval
         ,xval))))

- gensym creates and returns a new symbol table entry, guaranteed never to be the same as any other symbol

- Note that the calls to gensym are not part of the quoted and returned syntax
  - Evaluated, and their results used, at compile time

- Single evaluation of forms in max
  - C does not have a mechanism for statement-only features like storage allocation with an expression
  - Lisp does not distinguish between statements and expressions

Exercise 3.10. Scott, Exercise 3.23
3.3 Heap storage

The other end of memory

In the standard organization of memory, the stack grows from one end, the heap grows from the other

- The stack is organized FIFO
- The heap has no such time guarantees
- Allocations in the heap can vary in size, remain relevant for indeterminate periods

Simple heap management

Recall memory usage in the C/C++ family, or assembly language

- Declare specific data structures via `struct`, or a fixed multiple of size for an array
  - Very little in the way extending a data structure once declared
- One call `malloc` to allocate memory, another call `free` to release it
- Be wary of forgetting to free unused space!
- Be wary of keeping pointers into freed space!
- Fast and low overhead, but a high burden of error-prone space management on the individual application and programmer
- Problems of fragmentation — small, isolated free spaces separated by long-lived structures

Automatic garbage collection

In the 90s, automatic garbage collection became common

- Driven by higher-level (functional, object-oriented) academic languages showing feasibility
- Part of a trend of languages coming with larger and larger runtime systems and operating system links

Mark-and-scan garbage collection

- General idea: allocate heap space from the end of memory towards the stack
  - With each allocation, set aside extra bits for marks
- When the stack and heap collide (or when the heap hits a certain size), pause from executing program, and run garbage collector
- The garbage collector starts with pointers from registers and from the stack into the heap
- "Walks" the pointers, marking everything it finds as in use
- Then everything else must not still be in use, and can be re-used

Copying garbage collection

- General idea: divide the heap into two halves, allocate from only one half at a time
  - When that half fills, pause the program and run the garbage collector
- Again starting with live pointers from the heap and stack, copy live heap space from one half to the other half
  - Update pointers as they are walked
  - After copying resume the program, continuing to allocate from the half into which we just copied, until it fills and starts garbage collection again
- Can improve locality of reference, virtual memory performance
Generational garbage collection
Motivation: take advantage of the fact that space which has been used longer will probably also stay in use longer.

Divide the heap into generations, each of which is separately collected
• Older generations are collected less frequently
• Often combined with copy-collection — each generation in two parts, copying from one to the other

Reference counting
An appealing idea
• Every allocated chunk of memory has extra space set aside
• Like mark-scan, but space not used for marks
• Keep a count of the number of other places which point to it
• Circular structures can be a problem

4 Types
Why types?
• Provide context for operations
  – For example, to distinguish integer and floating-point addition
• Detect and prohibit nonsensical operations
• Documentation which is automatically checked for correctness
• Opportunities for the compiler to optimize performance
  – Because we don’t have to check cases at runtime
  – Or for example register allocation in the presence of pointers

Scalar and composite
• Scalar types are indivisible
  – Most built-in types: integers, booleans, characters
  – In many languages, enumerated types
• Composite types are data structures with several distinct components
  – Some built-in types: String in Java, for example
  – Arrays
  – Most user- and library-defined types

When are two types the same?
• Matters when passing parameters, making assignments.
• Two general ways to decide:
  – Decide based on structure
  – Decide based on their name
• Record types
Structural equivalence

- These should be considered the same:

```c
type R1 = struct {
  int a, b;
}
```

```c
type R2 = struct {
  int a;
  int b;
}
```

- What if the fields aren’t in the same order?

```c
type R3 = struct {
  int a;
  int b;
}
```

```c
type R4 = struct {
  int b;
  int a;
}
```

Many (but not all) languages say that these are structurally equivalent
- Once again, it is a choice for the language designer

Name equivalence

- If the name is the same, the type is the same
  - Rules out the R1, R2 equivalence of the previous slide.
- What about type aliases?

```c
typedef old_type new_type;
```

- Of course they should be interchangeable!

```c
typedef unsigned int mode_t;
```

- Of course they should not be interchangeable!

```c
typedef double degrees_fahrenheit;
typedef double degrees_celsius;
```

- Sometimes and sometimes not?

5 Functional programming and Haskell

5.1 Exercises on Haskell basics

**Exercise 5.1.** [Hutton Ex. 2.7.2] Correctly parenthesize these numeric expressions:

- $2^3 \times 4$
- $2 \times 3 + 4 \times 5$
- $2 + 3 \times 4^5$

**Exercise 5.2.** Keller and Chakravarty, [Sec. 1 (First Steps)] Ex. 1-3.
Exercise 5.3.  [Keller and Chakravarty] Which of the following identifiers can be function or variable names?

- square_1
- 1square
- Square
- square!
- =square='=

Exercise 5.4.  [Keller and Chakravarty] Define a new function showResult that, for example given the number 123, produces a string as follows:

```
showResult 123 ==> "The result is 123"
```

Use the function show in the definition of the new function.

Exercise 5.5.  [Includes items from Hutton] Which of these expressions are well-typed, and what types do those expressions have?

- ['a'. 'b'. 'c']
- ('a'. 'b'. 'c')
- ('a'. 'b'. 'c', 'a'. 'b'. 'c')
- ['a'. 'b'. 1]
- ('a'. 'b'. 1)
- [(False, '0'), (True, '1')]  
- [(False, True), ('0', '1')]
- ([False, True], ['0', '1'])
- ([False, '0'], [True, '1'])
- [tail, init, reverse]

Exercise 5.6.  Write Haskell definitions which have the following types.

- [(Int,Int)]
- Int -> Int -> Bool -> Int
- Char -> (Char, Char)
- Int -> (Int -> Int) -> Int

Exercise 5.7.  [Hutton Ex. 3.11.3] What types do these functions have? Try to work them out by hand before checking your answers in GHCI.

- second xs = head (tail xs)
- swap (x,y) = (y,x)
- pair x y = (x,y)
- double x = x*2
- twice f x = f (f x)
Exercise 5.8. Write a module LesserInt exporting a single function lesserInt which takes two integers, and returns the one which is lower in value.

To wrap your function in the module LesserInt, create a new file called LesserInt.hs whose first line is module LesserInt where, with your definition for lesserInt on its own line below.

Exercise 5.9. [Keller and Chakravarty] Write a function showAreaOfCircle which, given the radius of a circle, calculates the area of the circle,

\[
\text{showAreaOfCircle 12.3} \\
\implies \text{"The area of a circle with radius 12.3cm is about 475.291552615999 cm}^2\text{"}
\]

Use the show function, as well as the predefined value \( \text{pi} :: \text{Floating a} \Rightarrow a \) to write showAreaOfCircle.

Exercise 5.10. [Keller and Chakravarty] Write a function sort2,

\[
\text{sort2 :: Ord a} \Rightarrow a \rightarrow a \rightarrow (a, a)
\]

which accepts two Int values as arguments and returns them as a sorted pair, so that \( \text{sort2 5 3} \) is equal to \((3,5)\). How can you define the function using a conditional, how can you do it using guards?

Exercise 5.11. [Keller and Chakravarty] Define a module IsLower with a single function

\[
isLower :: \text{Char} \rightarrow \text{Bool}
\]

which returns \text{True} if a given character is a lower case letter. You can use the fact that characters are ordered, and for all lower case letters \( \text{ch} \) we have \( 'a' \leq \text{ch} \) and \( \text{ch} \leq 'z' \). Alternatively, you can use the fact that \(['a'..'z']\) evaluates to a list containing all lower case letters. Write your own version of isLower; do not use the standard version in Data.Char (or even import Data.Char).

Exercise 5.12. [Thompson] Write a module DoubleAll exporting one function doubleAll of type \([\text{Int}] \rightarrow [\text{Int}]\) which doubles each element of a list.

Exercise 5.13. [Thompson] Write a module Capitalize exporting one function capitalize which converts all lower-cases letters in its argument to upper-case letters. But leaves the other characters alone. The Haskell Data.Char library contains functions which will be useful here.

Exercise 5.14. [Thompson] Write a module CapitalizeOnly exporting one function capitalizeOnly which converts all lower-cases letter in its argument to upper-case letters, leaves upper-case letters alone, and removes other characters from the result. The Haskell Data.Char library contains functions which will be useful here.

Exercise 5.15. [Thompson] Write a module Matches exporting one function matches of type \(\text{Int} \rightarrow [\text{Int}] \rightarrow [\text{Int}]\) which returns all occurrences of the first argument in its second argument. So for example, \(\text{matches 10 [1,10,2,10,3,10,4]}\) returns \([10,10,10]\), and \(\text{matches 10 [11,14,17,21]}\) returns \([],\).

Exercise 5.16. [Keller and Chakravarty] Write a module Mangle exporting function mangle,

\[
mangle :: \text{String} \rightarrow \text{String}
\]

which removes the first letter of a word and attaches it at the end. If the string is empty, mangle should simply return an empty string:

\[
mangle \text{"Hello"} \implies \text{"elloH"} \\
mangle \text{"I"} \implies \text{"I"} \\
mangle \text{""} \implies \text{""}
\]
Exercise 5.17. [Keller and Chakravarty] Write a module Divider with a function dividedBy which implements division on Int,

\[
dividedBy :: \text{Int} \to \text{Int} \to \text{Int}
\]

by first writing a helper function that returns all the multiples of a given number up to a specific limit, and then using list functions on the resulting list.

\[
dividedBy 5 10 \implies 2 \\
dividedBy 5 8 \implies 1 \\
dividedBy 3 10 \implies 3
\]

Exercise 5.18. [Keller and Chakravarty] Define a module LengthTaker with the function length,

\[
\text{length} :: [\text{a}] \to \text{Int}
\]

It is quite similar to sum and product in the way it traverses its input list. Since length is also defined in the Haskell standard Prelude, hide it by adding the line

\[
\text{import Prelude hiding (length)}
\]

to your module.

Exercise 5.19. [Hutton Ex. 4.8.1, with solution] Use Haskell library functions to define a function halve,

\[
\text{halve} :: [\text{a}] \to ([\text{a}],[\text{a}])
\]

Exercise 5.20. [Hutton Ex. 4.8.2, with solution] Define a module Third exporting a single function third,

\[
\text{third} :: [\text{a}] \to \text{a}
\]

which returns the third element in a list, a) Using head and tail. b) Using list indexing !!. c) Using pattern matching.

Exercise 5.21. Write a module LastItem exporting the function lastItem, which returns the last item in a list

Exercise 5.22. Write a module LastButOne exporting the function lastButOne, which returns the next-to-last item in a list

Exercise 5.23. [Keller and Chakravarty] Write a module CountOdds exporting a recursive function countOdds which calculates the number of odd elements in a list of Int values:

\[
\text{countOdds} \ [1, 6, 9, 14, 16, 22] = 2
\]

Hint: You can use the Prelude function odd :: Int \to Bool, which tests whether a number is odd.

Exercise 5.24. [Keller and Chakravarty] Write a module RemoveOdd exporting a recursive function removeOdd that, given a list of integers, removes all odd numbers from the list, e.g.,

\[
\text{removeOdd} \ [1, 4, 5, 7, 10] = [4, 10]
\]

Exercise 5.25. Write the function isPalindrome, which checks if a list is a palindrome, the same backwards as forwards

Exercise 5.26. Write a module NeighborDups exporting a function noNeighborDups, which returns a list with consecutive duplicates removed
Exercise 5.27. Write a module `EncodeDecode` which exports the function `lengthEncode`, for example,

```haskell
lengthEncode "Aaabbcddeeeabb"
==>> [(1,'A'), (2,'a'), (3,'b'), (1,'c'),
     (2,'d'), (3,'e'), (1,'a'), (2,'b')]
```

Exercise 5.28. Extend your module `EncodeDecode` of Exercise 5.27 with the function `lengthDecode`, opposite of the above.

Exercise 5.29. Write a model `ListSplitter` exporting the function `(splitListAt n xs)`, which splits a list into two lists, the first one with `n` elements.

Exercise 5.30. Consider these declarations:

```haskell
infixl 5 'test1'
infixl 7 'test2'
```

Complete the definition of `test1` and `test2` with two function declarations — it doesn’t matter what they do, just make them distinct enough for you to tell the difference between them as easily as you could tell the difference between other operators like addition and multiplication.

How do `test1` and `test2` behave differently with respect to each other? In a series of several applications of each?

Vary the declarations to use `infixr` and `infix` instead of `infixl`, and to use various different numbers. How does this change how the operators behave?

5.2 Functional datatypes

5.2.1 Algebraic data types

Algebraic data types

Haskell declares new data types with the `data` declaration

```haskell
data TYPENAME = CONSTRUCTOR1 ArgType1-1 ... ArgType1-n
             | CONSTRUCTOR2 ArgType2-1 ... ArgType2-m
             ...
             | CONSTRUCTORk ArgTypek-1 ... ArgTypek-l
```

- The `List` type is the same idea, just with special syntax

Pattern matching

All data types can be `pattern-matched` in a function definition or case structure

```haskell
data Season = Winter | Spring | Summer | Fall

isFall Fall = True
isFall _ = False
```

- Similarly for lists and for built-in enumerated types like `Int`
Exercise 5.31.  [Keller and Chakravarty] Write a module `Days` which exports:

- The definition of `Day` from [this page]. Module `Days` should export both the name of the type, and the names of its constructors.
- A function which, given a day, returns the data constructor representing the following day:

```haskell
nextDay :: Day -> Day
```

Exercise 5.32.  [Thompson] Write a module `MonthsAndSeasons` which exports a type `Month` as an algebraic type for the twelve months (use the full name of the month as constructors, and export both the type and constructor names), and a function `monthSeason` which maps a month to its member of the type `Season`.

```haskell
data Season = Winter | Spring | Summer | Fall
```

Exercise 5.33.  [Thompson] Consider a module `Shapes` with this type of geometric shapes,

```haskell
data Shape = Circle Float
            | Rectangle Float Float
```

encapsulating a value for the radius of a circle, or the dimensions of a rectangle.

1. Add functions `area` and `perimeter` which take a `Shape` as an argument, and return the value of the respective property of that shape.
2. Add a constructor `Triangle` to `Shape` for triangles. The new constructor should take three `Float` values, the length of the sides of the triangle.
3. Add cases to `area` and `perimeter` for `Triangle`.

Exercise 5.34.  [Keller and Chakravarty] How would you define a data type to represent the different cards of a deck of poker cards? How would you represent a hand of cards?

Define a function `value21` which, given a hand of cards calculates its values according to the 21- (Blackjack) rules: that is, all the cards from 2 to 10 are worth their face value. Jack, Queen, King count as 10. The Ace card is worth 11, but if this would mean the overall value of the hand exceeds 21, it is valued at 1.

Exercise 5.35.  The standard functions `head` and `tail`,

```haskell
head :: [a] -> a
tail :: [a] -> [a]
```

are partial. a) [Keller and Chakravarty] Implement total variants `safeHead` and `safeTail` by making use of `Maybe` in the function results. b) [Hutton Ex. 4.8.3 with solution] Implement `safeTail` to return an empty list where `tail` returns an error,

- Using a conditional expression
- Using guarded equation
- Using pattern matching.

Exercise 5.36.  [Keller and Chakravarty] Write a function `myLength`

```haskell
myLength :: [a] -> Int
```

that, given a list `l`, returns the same result as `length l`. However, implement `myLength` without any explicit pattern matching on lists; instead, use the function `safeTail` from the previous exercise to determine whether you reached the end of the list and to get the list tail in case where the end has not been reached yet.

31
List comprehension notation

Express one list in terms of other lists

*Prelude> [ 2*x | x <- [1,2,3] ]
[2,4,6]
*Prelude> [ (x,y) | x <- [1,2,3], y <- ['a', 'b', 'c'] ]
[(1,'a'),(1,'b'),(1,'c'),(2,'a'),(2,'b'),
 (2,'c'),(3,'a'),(3,'b'),(3,'c')]
*Prelude> [ x | x <- [1..10], x `mod` 3 == 1 ]
[1,4,7,10]

Exercise 5.37. Use list comprehension notation to complete this function definition to take a list of integers, and return a list containing only the elements of the argument which are divisible by three:

\[
\text{dividesByThree} :: \text{[Int]} \rightarrow \text{[Int]}
\]
\[
\text{dividesByThree} \ a = [ x \mid x \in a, x \mod 3 == 1 ]
\]

Exercise 5.38. Use list comprehension notation to write the function \( \text{capVowelsFirst} \) that takes a list of strings, and return a list containing only the elements of the argument which start with a capital vowel.

5.2.2 Leftist heaps

Trees and heaps

Lovely memories from the simpler days of CS340

- A tree is a structure which can either be
  - Empty, or
  - A node, with a value plus subtrees

In a binary tree, every node has two subtrees

- A heap, generally speaking, is a structure used for finding and deleting minimum elements
  - Often implemented through an array, or as a binary tree
  - The value at any node is no larger than the values at either child

Tree rank

The rank of a tree node is the length of its right spine
Exercise 5.39. Write a Haskell data type BlackWhiteTree with two constructors

- Black, which takes two BlackWhiteTree arguments
- White, which takes no arguments

Encode each of the above examples (plus the one below) as BlackWhiteTree values with names bw1, bw2, etc.

Exercise 5.40. For your BlackWhiteTree type of Exercise 5.39 write a function bwNodeRank which returns the rank of the top node of a BlackWhiteTree.

The leftist property

A tree has the leftist property when the rank of any left child is at least as big as the rank of its right sibling.

A leftist heap is a heap based on a binary tree with the leftist property.

- We will see that leftist heaps support a nice merging behavior.

Exercise 5.41. For your BlackWhiteTree type of Exercise 5.39 write a function bwHasLeftist which returns True when given the root node of a tree with the leftist property.

A leftist heap of floating point values

Let’s design a leftist heap LDH for holding floating-point values.

The heap should support these operations:

- emptyHeap :: LDH
- isEmpty :: LDH -> Bool
- insert :: Double -> LDH -> LDH
- merge :: LDH -> LDH -> LDH
- findMin :: LDH -> Double
- deleteMin :: LDH -> Double

The last four functions should all preserve both heap ordering and the leftist property.

Standard trick: store the rank

- To speed comparisons, we store the rank in each node
- To better guarantee properties, we restrict access to the constructors
module LeftistDoubleHeap (LDH, emptyHeap, isEmpty, 
insert, merge, 
findMin, deleteMin) where

data LDH = EmptyLDH | NodeLDH Int Double LDH LDH

Some helper functions

We will need the rank of nodes

leftistRank :: LDH -> Int
leftistRank EmptyLDH = 0
leftistRank (NodeLDH n _ _ _) = n

Assemble a NodeLDH so that we satisfy the leftist property

makeLDH :: Double -> LDH -> LDH -> LDH
makeLDH e h1 h2 = let r1 = leftistRank h1
                            r2 = leftistRank h2
                           in if r1 >= r2
                                 then NodeLDH (1+r2) e h1 h2
                                 else NodeLDH (1+r1) e h2 h1

The easy ones

Returning and testing for an empty tree is straightforward

emptyHeap :: LDH
eemptyHeap = EmptyLDH

isEmpty :: LDH -> Bool
isEmpty EmptyLDH = True
isEmpty _ = False

Merging two heaps

The main decision in merging two trees is picking the smaller of the two top elements to be the new top element

- We merge the right spines in the same way that we can merge sorted lists
- Since the right spine is never longer than the left spine, we are assured of $O(\log n)$ merging
- The makeLDH helper assures that the leftist property is upheld

merge :: LDH -> LDH -> LDH
merge EmptyLDH h = h
merge h EmptyLDH = h
merge h1@(NodeLDH _ e1 11 r1) h2@(NodeLDH _ e2 12 r2) = 
  if e1<e2
      then makeLDH e1 11 (merge r1 h2)
      else makeLDH e2 12 (merge h1 r2)

Merging example

merge (NodeLDH 3 1.1 aa (NodeLDH 2 3.0 bb (NodeLDH 1 5.4 cc EmptyLDH)))
   (NodeLDH 2 2.0 dd (NodeLDH 1 4.2 ee EmptyLDH))
   ===>
   if 1.1 < 2.0
      then makeLDH 1.1 aa (merge (NodeLDH 2 3.0 bb (NodeLDH 1 5.4 cc EmptyLDH))
                                     (NodeLDH 2 2.0 dd (NodeLDH 1 4.2 ee EmptyLDH)))

else makeLDH 2.0 dd
    (merge (NodeLDH 3 1.1 aa (NodeLDH 2 3.0 bb (NodeLDH 1 5.4 cc EmptyLDH)))
     (NodeLDH 1 4.2 ee EmptyLDH))

==> makeLDH 1.1 aa (merge (NodeLDH 2 3.0 bb (NodeLDH 1 5.4 cc EmptyLDH))
    (NodeLDH 2 2.0 dd (NodeLDH 1 4.2 ee EmptyLDH)))

==> makeLDH 1.1 aa
    (if 3.0<2.0
     then makeLDH 3.0 bb (merge (NodeLDH 1 5.4 cc EmptyLDH)
      (NodeLDH 2 2.0 dd (NodeLDH 1 4.2 ee EmptyLDH)))
     else makeLDH 2.0 dd (merge (NodeLDH 2 3.0 bb (NodeLDH 1 5.4 cc EmptyLDH))
      (NodeLDH 1 4.2 ee EmptyLDH)))

==> makeLDH 1.1 aa
    (makeLDH 2.0 dd (merge (NodeLDH 2 3.0 bb (NodeLDH 1 5.4 cc EmptyLDH))
     (NodeLDH 1 4.2 ee EmptyLDH)))

==> makeLDH 1.1 aa (makeLDH 2.0 dd
    (if 3.0 < 4.2
     then (makeLDH 3.0 bb (merge (NodeLDH 1 5.4 cc EmptyLDH)
      (NodeLDH 1 4.2 ee EmptyLDH)))
     else (makeLDH 4.2 bb (merge (NodeLDH 2 3.0 bb (NodeLDH 1 5.4 cc EmptyLDH))
      EmptyLDH))))

==> makeLDH 1.1 aa (makeLDH 2.0 dd
    (makeLDH 3.0 bb (merge (NodeLDH 1 5.4 cc EmptyLDH) (NodeLDH 1 4.2 ee EmptyLDH))))

==> makeLDH 1.1 aa (makeLDH 2.0 dd (makeLDH 3.0 bb
    (if 5.4 < 4.2
     then (makeLDH 5.4 cc (merge EmptyLDH (NodeLDH 1 4.2 ee EmptyLDH)))
     else (makeLDH 4.2 ee (merge (NodeLDH 1 5.4 cc EmptyLDH) EmptyLDH))))

==> makeLDH 1.1 aa (makeLDH 2.0 dd (makeLDH 3.0 bb (makeLDH 4.2 ee
    (merge (NodeLDH 1 5.4 cc EmptyLDH) EmptyLDH)))

==> makeLDH 1.1 aa (makeLDH 2.0 dd (makeLDH 3.0 bb (makeLDH 4.2 ee
    (NodeLDH 1 5.4 cc EmptyLDH)))

Insertion and deletion can just use heap merging

insert :: Double -> LDH -> LDH
insert e h = merge (NodeLDH 1 e EmptyLDH EmptyLDH) h

findMin :: LDH -> Double
findMin EmptyLDH = error "Reading from empty heap"
findMin (NodeLDH _ e _ _) = e

deleteMin :: LDH -> LDH
deleteMin EmptyLDH = error "Deleting from empty heap"
deleteMin (NodeLDH _ _ h1 h2) = merge h1 h2

Exercise 5.42. Assemble the module LeftistDoubleHeap, and define several example heaps, extracting information from each.

References

5.2.3 Red-black trees

Red-black trees
A balanced tree has the same number of elements and the same depth on the left- and right-sides of every node

• Balanced trees guarantee \(O(\log n)\) operations in many cases

• True balance can be expensive to maintain, so a number of algorithms allow us to approximate balance more cheaply

• Red-black trees are an approximation to balanced trees
  – Not perfectly balanced, but close enough

Starts with an ordered binary tree
• No duplicate elements, modeling a set

Adds a color, red or black, to each node of the tree
• Leaves are considered black

Plus two invariants about the structure of the tree:
1. All paths from the root of the tree to an empty leaf must have the same number of black nodes
2. No red node has a red child

Red-black tree data type
We’ll design an implementation for red-black trees holding floating-point values

• Could also model the color with a \texttt{Bool} field, for example \texttt{True} for black and \texttt{False} for red

module RedBlackDoubleTree (RBDT, emptyTree, isEmpty, member, insert) where
data Color = Red | Black
data RBDT = EmptyRBDT
            | BranchRBDT Color RBDT Double RBDT

Exercise 5.43. Write a function \texttt{verifyRBinvariants} which checks that an RBDT value satisfies the invariants that

1. Its numbers come in order
2. No red node has a red child
3. Every path from the root to an empty leaf has the same number of black nodes

Make sure that your function traverses the tree only once, and does not re-descend to re-count the black nodes at every branch (there is a hint for this last requirement on page 58).
Basic operations

- Empty trees are straightforward

  ```haskell```
  emptyTree :: RBDT
  emptyTree = EmptyRBDT

  isEmpty :: RBDT -> Bool
  isEmpty EmptyRBDT = True
  isEmpty _ = False
  ```haskell```

- Checking for membership is just as in any ordered binary tree

  ```haskell```
  member :: Double -> RBDT -> Bool
  member _ EmptyRBDT = False
  member e (BranchRBDT _ lt e0 rt) =
    case compare e e0 of
      LT -> member e lt
      EQ -> True
      GT -> member e rt
  ```haskell```

Top-level insertion

We will adopt the helpful convention that our trees will always have a black root node, even if top-level manipulations end with a red root

```haskell```
insert e t =
  case helper e t of
    (BranchRBDT _ lt e0 rt) -> BranchRBDT Black lt e0 rt
    _ -> error "Internal error"
```

Insertion and balancing

- Superficially the helper looks like recursive colorless sorted-tree insert, but does extra work to preserve the invariants

  ```haskell```
  helper :: Double -> RBDT -> RBDT
  ```haskell```

- The helper returns a singleton tree when it reaches an empty leaf

  ```haskell```
  helper e EmptyRBDT = BranchRBDT Red EmptyRBDT e EmptyRBDT

To preserve the number of black nodes on each path, the new node is red

- But this may give us a red node with a red child

  ```haskell```
  helper e tt@(BranchRBDT cl lt e0 rt) =
    if e<e0
      then balance cl (helper e lt) e0 rt
    else if e>e0
      then balance cl lt e0 (helper e rt)
    else tt

So we apply a separate balance function instead of the BranchRBDT constructor to check for violations of the red-red invariant

37
When the helper breaks the red-red invariant

def helper

• adds a new bottommost node

balance Black a x
(BranchRBdT Red (BranchRBdT Red b y c) z d) =
BranchRBdT Red (BranchRBdT Black a x b)
y (BranchRBdT Black c z d)

balance must find the bad pattern

• When using a subtree with a red root which has a red child

• Which can only happen if parent node is black

• Rearrange the tree to restore the invariants

• Push the forbidden red-red pair upwards

Four cases which balance must find

balance :: Color -> RBDT -> Double -> RBDT -> RBDT

balance Black (BranchRBdT Red (BranchRBdT Red a x b) y c) z d =
BranchRBdT Red (BranchRBdT Black a x b) y (BranchRBdT Black c z d)
balance Black (BranchRBdT Red a x (BranchRBdT Red b y c)) z d =
BranchRBdT Red (BranchRBdT Black a x b) y (BranchRBdT Black c z d)
balance Black a x (BranchRBdT Red (BranchRBdT Red b y c) z d) =
BranchRBdT Red (BranchRBdT Black a x b) y (BranchRBdT Black c z d)
balance Black a x (BranchRBdT Red b y (BranchRBdT Red c z d)) =
BranchRBdT Red (BranchRBdT Black a x b) y (BranchRBdT Black c z d)

balance color left root right = BranchRBdT color left root right

• Can you see now the two reasons why we always set the color of the final root node to black?

Exercise 5.44. Assemble the module RedBlackDoubleTree, and define several example trees, extracting information from each. Use your function verifyRBinvariants from Exercise 5.43 to make sure it behaves correctly.

Exercise 5.45. We can optimize this code slightly based on the way helper knows whether its recursive call is in the left or right subtree. Replace balance with two functions balanceLeft and balanceRight, which check for a red-red violation only in the left or right subtree, respectively. Then update helper to call the appropriate replacement for balance. Use your function verifyRBinvariants from Exercise 5.43 to make sure it behaves correctly.

References

5.2.4 Huet’s Zipper

Locations within a tree

Sometimes we need to discuss not just a tree (or other structure) but a particular subtree of the overall structure

- For example, to visually navigate a structure
  - Moving left and right, up and down
  - Possibly editing along the way
- We need to separate one subtree from its context
- The zipper is a technique for implementing this shifting view
- Intuitively, the technique peels up part of a structure, as if turning a glove inside-out when removing it from your hand

We will work on a general tree

```haskell
data Tree = Branch [Tree]
           | Leaf Double
```

It is unusual to have no values at branches, but it will simplify the presentation

What is a context?

If we grab on to the link between a branch and one of its child trees, what is the context that we find on the other end from that branch?

- Siblings to its left
- Siblings to its right
  - The siblings are just trees
- More context above
  - Up to the root node
- We can encode the context as a data type

```haskell
data Context = Root
             | Siblings [Tree] Context [Tree]
```

- Then a tree with a particular subtree highlighted is just a pair of this context and the subtree

```haskell
data Location = Loc Context Tree
```

Moving around within a node

What does it mean to "navigate" from a node to one of its siblings?

For example, if we "move right"

- The first sibling to the right becomes the subtree of interest
- The previous subtree of interest becomes a new sibling to the left

```haskell
goRight (Loc (Siblings ls p (r:rs)) t) = Loc (Siblings (t:ls) p rs) r
goRight _ = error "Cannot go right"
```

And similarly for moving left
goLeft (Loc (Siblings (1:ls) p rs) t) = Loc (Siblings ls p (t:rs)) l

- Note that the left siblings are stored with the nearest first
  - So reversed from a left-to-right ordering

Moving up

What about moving up in the tree?

- The previous subtree of interest, plus the siblings of the current context, become part of a new branch node of interest
- The context above the old context becomes the new context

goUp (Loc (Siblings ls p rs) t) = Loc p (Branch (pushOnto ls (t:rs)))

where pushOnto [] xs = xs
    pushOnto (y:ys) xs = pushOnto ys (y:xs)
goUp _ = error "Cannot go up"

Descending into a child node

- Its first child becomes the current subtree
- Its other siblings, plus the old context above, become the new context

goDown (Loc p (Branch (t:ts))) = Loc (Siblings [] p ts) t

No descending into a leaf, or an empty branch

- We identify with the link between parent and child, and there are no links below a leaf

Adding a subtree

We can make changes to the tree as we navigate it

- Since we reassemble tree and context structure as we go, we do not need to change nonlocal structures

Moving to the left or right is straightforward

insertLeft (Loc (Siblings ls p rs) d) t = Loc (Siblings (t:ls) p rs) d

insertRight (Loc (Siblings ls p rs) d) t = Loc (Siblings ls p (t:rs)) d

When we insert into the branch below, the new subtree becomes the current focus

insertBelow (Loc p (Branch sibs)) t = Loc (Siblings [] p sibs) t

Removing the current subtree

Removing is a little complicated, because if we remove the current subtree then we also need to move

- We need to pick default directions
- Move right if possible, else try left, else try up

prune (Loc (Siblings ls p (r:rs)) _) = Loc (Siblings ls p rs) r
prune (Loc (Siblings (1:ls) p []) _) = Loc (Siblings ls p []) l
prune (Loc (Siblings [] p []) _) = Loc p (Branch [])
prune (Loc Root _) = error "Cannot prune root node"
Derivatives
How do we think about zippers for arbitrary data types?

- The intuition comes from the derivatives of calculus
- \( d(u+v) = du + dv \)
- \( d(uv) = udv + vdu \)
- Multiplication is analogous to gathering data together in a tuple or with a constructor
- Addition is analogous to the alternative constructors allowed for a data type
- So the context associated with a pair would be either
  - A regular left element, plus a context to the right; or
  - A context to the left, and a regular right element
- And if a type can have one of two forms, then contexts over that type will also have one of two forms

Exercise 5.46. Extend the general trees and contexts of this section to have a value associated not just with the leaves, but with branches as well.

Exercise 5.47. Develop a notion of contexts for binary trees.

References


5.2.5 Laziness in data structures

Haskell evaluation does as little as possible
Whenever a function doesn’t require some part of an argument, it won’t necessarily be evaluated.

secondOfList (_:x:_ ) = x

If we apply this function to a list with errors, we won’t trigger these errors if the second element of the list doesn’t have errors.

*Prelude> secondOfList (error "e1" : 3 : error "rest of the list")
3

Unbounded lists

- We can describe lists which are arbitrarily long
- If we do something that requires the whole list (like displaying it, or foldl), then of course our program will not terminate productively
- But if we use a function like take or takeWhile then we can draw only as much as we need
Specifying an unending list

- Simple cases

```haskell
onesForever = 1 : onesForever
twoAndUp = [2..]
fromThreeByFives = [3,8..]
byTens = byTens' 10 where byTens' x = x : byTens' (10+x)
```

**Exercise 5.48.** Complete the recursive function `genFibs` in the style of `byTens` so that we could use `genFibs` to define a list corresponding to the Fibonacci numbers.

```haskell
genFibs :: Int -> Int -> [Int]
genFibs n1 n2 = n1 : -- FILL IN HERE
```
The first argument `n1` corresponds to the present "first" element of the list of numbers, and the two arguments to the recursive call should lead to subsequent elements.

Prelude> take 6 (genFibs 10 100)
[10,100,210,320,530]

**Exercise 5.49.** Use `genFibs` from the previous exercise to define the list `fibs` of Fibonacci numbers.

Prelude> take 10 fibs
[0,1,1,2,3,5,8,13,21,34]

The Sieve of Eratosthenes

- Write out the numbers we’re interested in testing for primality

```
  1  2  3  4  5  6  7  8  9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99 100
```

The Sieve of Eratosthenes

- Write out the numbers we’re interested in testing for primality
  - 1 is not a prime, so scratch it out
The Sieve of Eratosthenes

• Write out the numbers we’re interested in testing for primality
  • 1 is not a prime, so scratch it out
  • Look at the lowest unmarked number — mark it as prime

The Sieve of Eratosthenes

• Write out the numbers we’re interested in testing for primality
  • 1 is not a prime, so scratch it out
  • Look at the lowest unmarked number — mark it as prime
  • But strike out its multiples — they’re definitely not prime

The Sieve of Eratosthenes

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- And so on with the new lowest unmarked number, and so on, and so on, and so on

\[ \begin{array}{cccccccccc}
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21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\
31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 \\
51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 \\
61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 \\
71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 \\
81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 \\
91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 \\
\end{array} \]

- So let’s code that up

Coding up the sieve

- One pass of the sieving algorithm:
  - Accept the first element as prime
  - Remove all multiples of the first element from the rest of the list
  - Sieve what’s left

\[
\text{sieve} \ (x:xs) = x : \text{sieve} \ [ z \mid z \not< xs, z \mod x /= 0 ]
\]

- Then the list of all prime numbers is just

\[
\text{primes} = \text{sieve} \ [2..]
\]

- So long as we don’t try to find the last element of the list!

*Prelude*> take 30 primes

*Prelude*> head (filter (> 10000) primes)
10007

So how do the various list functions work with nonterminating lists?

- Recall \textit{foldl} and \textit{foldr}.

\[
\text{foldl} \ f z \ [ ] = z \\
\text{foldl} \ f z \ (x:xs) = \text{foldl} \ f \ (f z x) \ xs
\]

\[
\text{foldr} \ f z \ [ ] = z \\
\text{foldr} \ f z \ (x:xs) = f x \ (\text{foldr} \ f z \ xs)
\]
• What if we want to apply a folding function to a list like `primes`?
  – (Say, with an \( f \) that constructs some other data structure)
• `foldl` will try to deconstruct the list all the way to the end before returning anything else
• If \( f \) returns some data constructor, then `foldr` can avoid trying to traverse the whole list

5.3 Higher-order functions

5.3.1 Functions as values

5.3.2 Patterns of recursion over lists: filter, map, fold

Finding patterns

```haskell
sumOneTo :: Int -> Int
sumOneTo x | x>0 = x + sumOneTo (x-1)
sumOneTo _ = 0

prodOneTo :: Int -> Int
prodOneTo x | x>0 = x * prodOneTo (x-1)
prodOneTo _ = 1
```

**Exercise 5.50.** [Keller and Chakravarty] Rewrite the definition of `mapInts`

```haskell
mapInts :: (Int -> Int) -> [Int] -> [Int]
mapInts f [] = []
mapInts f (x : xs) = f x : map f xs
```

to use case notation. That is, complete the following definition

```haskell
mapInts :: (Int -> Int) -> [Int] -> [Int]
mapInts f xs = case xs of
    ... 
```

**Exercise 5.51.** Write `intTreeFoldl` and `intTreeFoldr`, folding functions on integer binary trees,

```haskell
data IntTree = Branch IntTree IntTree
            | Leaf Int
```

The functions should have signatures

```haskell
intTreeFoldl :: (t -> Int -> t) -> t -> IntTree -> t
intTreeFoldr :: (Int -> t -> t) -> t -> IntTree -> t
```

and should apply the operations and default with the given associativity of the values in the leaves. For example,

```haskell
*Main> let t1 = (Branch (Branch (Leaf 10) (Leaf 3))
            (Branch (Leaf 2) (Leaf 1)))
*Main> intTreeFoldl (\x y -> x-y) 0 t1
-16
*Main> intTreeFoldr (\x y -> x-y) 0 t1
8
```
Exercise 5.52. Write `binaryTreeFoldl` and `binaryTreeFoldr`, folding functions on general binary trees,

```haskell
data BinaryTree a = Branch (BinaryTree a) (BinaryTree a)
                 | Leaf a
```

Exercise 5.53. Write `binaryTreeFilter`, of type `(a -> Bool) -> (BinaryTree a) -> Maybe (BinaryTree a)`. It must be `Maybe`, since we have no constructor for an empty tree.

5.3.3 More patterns of recursion over lists

5.4 Polymorphism and type classes

5.4.1 Polymorphic functions

Polymorphic functions

- In functional languages, you’ll see *polymorphic* more often than *generic*
- Polymorphic functions

```haskell
  a -> a
  b -> (a -> b -> c) -> a -> c
  (([Char], a) -> b) -> [Char] -> (a -> b)
```

- What are these `a`, `b`, `c`?
  - *Type variables*
  - Quantified outermost, so `t -> t` means `∀ t. (t -> t)`
  - Write type variables with an initial lower-case letter

Benefits of generic types

- Avoid casts from e.g. the collections library
- Detect and prohibit *further* nonsensical operations
- *Finer-grained* documentation which is automatically checked for correctness
- Reduce code duplication
- More easily distinguish bugs in using a library from bugs within a library

Collections classes

- Best-known user of generics
  - `Set<A>, Map<A,B>, List<A>`
5.4.2 Polymorphic data types

Data types can be polymorphic too

```haskell
data BinaryTree a = Branch (BinaryTree a) (BinaryTree a) 
  | Leaf a
```

```haskell
binaryTreeMap f (Leaf x) = Leaf (f x)
binaryTreeMap f (Branch t1 t2) 
  = Branch (binaryTreeMap f t1) (binaryTreeMap f t2)
```

- What is the type of the empty list?

**Exercise 5.54.** [Keller and Chakravarty] The map function is just a special case of foldr. Can you rewrite the map definition in terms of foldr? Complete the following definition:

```haskell
map :: (a -> b) -> [a] -> [b]
map f = foldr ...
```

**Exercise 5.55.** Use a fold function to implement the exclusive-or function xor of type [Bool] -> Bool, which returns True when there is exactly an odd number of True values in the list.

**Exercise 5.56.** Use a fold function to concatenate a list of lists together into a single list,

```haskell
concatAll :: [[t]] -> [t]
```

**Exercise 5.57.** Use filter to write a function that removes the vowels from a string.

**Exercise 5.58.** Redefine length and reverse using the fold functions.

**Exercise 5.59.** For all of the functions with fold, which are more efficient with foldr, and which are more efficient with foldl?

**Exercise 5.60.** [Keller and Chakravarty] Rewrite the definition of map

```haskell
map :: (a -> b) -> [a] -> [b]
map f [] = []
map f (x : xs) = f x : map f xs
```

to use case notation. That is, complete the following definition

```haskell
map f xs = case xs of
  ...
```

5.4.3 Type classes

Declaring classes

For example:

```haskell
class Eq a where
  (==) :: a -> a -> Bool
  (/=) :: a -> a -> Bool
  x == y = not (x /= y)
  x /= y = not (x == y)
```

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• Must give type signatures, since we’re specifying operations without (necessarily) giving an implementation

• Can also give a default implementation

  – Notice here that we define the two operations in terms of each other!
  – So the instance declaration must define at least one of the two operations — otherwise the definitions make no sense

Exercise 5.61. Adapt your module LeftistDoubleHeap from Exercise 5.44 as simply LeftistHeap, with type LH polymorphic in the type of element which it contains. Since merge makes a comparison on elements of the contained type, most of the exported functions will need an Ord a constraint,

emptyHeap :: Ord a => LH a
isEmpty :: LH a -> Bool
insert :: Ord a => a -> LH a -> LH a
merge :: Ord a => LH a -> LH a -> LH a
findMin :: Ord a => LH a -> a
deleteMin :: Ord a => LH a -> LH a

Exercise 5.62. Adapt your module RedBlackDoubleTree from Exercise 5.42 as simply RedBlackTree, with type RBT polymorphic in the type of element which it contains. Since helper and member make a comparison on elements of the contained type, most of the exported functions will need an Ord a constraint,

emptyTree :: Ord a => RBT a
isEmpty :: RBT a -> Bool
member :: Ord a => a -> RBT a -> Bool
insert :: Ord a => a -> RBT a -> RBT a
helper :: Ord a => a -> RBT a -> RBT a
balance :: Color -> RBT a -> a -> RBT a -> RBT a

Exercise 5.63. [Keller and Chakravarty] Implement a function deleteSorted,

deleteSorted :: Ord a => a -> [a] -> [a]

which removes a value passed as first argument from a sorted list given as the second argument. If the value does not occur in the list, the list is returned unchanged. Exploit the fact that the list is sorted: if an element is not present in the list, stop the search as early as possible.

Exercise 5.64. Change the declaration of instance Show MyComplex so that the real coefficient is not shown when it is zero and the complex coefficient is non-zero.

Exercise 5.65. Complete the declaration of instance Show WordInt to print any integer as words.

Another example - complex numbers

There’s a built-in class of complex numbers, but let’s make our own

data MyComplex = MyComplex Double Double

instance Show MyComplex where
  show (MyComplex x iy) = case compare iy 0 of
    GT -> show x ++ "+" ++ show iy ++ "i"
    EQ -> show x
    LT -> show x ++ "-" ++ show (-iy) ++ "i"

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instance Num MyComplex where
  (MyComplex x iy) + (MyComplex x' iy') = MyComplex (x + x') (iy + iy')
  (MyComplex x iy) - (MyComplex x' iy') = MyComplex (x - x') (iy - iy')
  (MyComplex x iy) * (MyComplex x' iy') = MyComplex (x*x' - iy*iy')
      (x*iy' + x*iy')

abs (MyComplex x iy) = MyComplex (sqrt (x*x + iy*iy)) 0
signum (num@(MyComplex x iy)) = let (MyComplex a _) = abs num
  in MyComplex (x/a) (iy/a)
fromInteger n = MyComplex (fromInteger n) 0.0

More complex operations

instance Fractional MyComplex where
  (MyComplex x iy) / (MyComplex x' iy')
  = let denom = x'*x' + iy'*iy'
      in MyComplex ((x*x' + iy*iy')/denom) ((x'*iy - x*iy')/denom)
recip (MyComplex x' iy')
  = let denom = x'*x' + iy'*iy'
      in MyComplex (x'/denom) (-iy'/denom)
fromRational n = MyComplex (fromRational n) 0.0

- Could define trigonometric etc. operations for Floating.

Instances of a polymorphic datatype

Two ways:

1. Declare the full type to be an instance, possibly with constraints on the type variables
2. Declare the type constructor to be an instance of class on higher kinds

Exercise 5.66. Adapt your module LeftistHeap from Exercise 5.61 to separate the functions into a class definition of heaps and heap operations, with type LH being one instance of the class. Use the approach of higher-kinded classes so that heaps are polymorphic.

Exercise 5.67. Adapt your module RedBlackTree from Exercise 5.62 to separate the functions into a class definition of trees and tree operations, with type RBT being one instance of the class. Use the approach of higher-kinded classes so that trees are polymorphic.

Constraints on an instance declaration

instance Eq a => Eq (BinaryTree a) where
  (Leaf x) == (Leaf y) = (x==y)
  (Branch xs1 xs2) == (Branch ys1 ys2) = xs1 == ys1 && xs2 == ys2
_ == _ = False

instance Show a => Show (BinaryTree a) where
  show (Leaf x) = "(Leaf " ++ show x ++ ")"
  show (Branch xs1 xs2)
    = "(Branch " ++ show xs1 ++ ", " ++ show xs2 ++ ")"

- Like constraints on a type signature
There are two more ways of defining a new type in Haskell

- One way is a type synonym, keeping equivalence
- Other way is doesn’t preserve interchangeability

Equivalent synonyms via type

type defines an abbreviation for our convenience

- The prelude defines String this way:

  ```
type String = [Char]
  ```

  We can use String in our type or instance declarations, but ghci can’t always echo the name back to us

- We can give type variables for polymorphic types as well:

  ```
type ListOfTuplesWithInt a = [(a,Int)]
  ```

  ```
tupWith1 :: ListOfTuplesWithInt Bool
  tupWith1 = [(True, 3), (False, 4), (False, 5)]
  ```

Distinct synonyms via newtype

newtype defines a type synonym which is not interchangeable with the original

- newtype DifferentTuple = DiffTup (Int, String)

- We use it as if it had been declared with data

  ```
intFromDiffTup (DifferentTuple (n,_)) = n
  ```

- But there are important differences with data
  - There can be only one constructor form
  - That constructor can have only one value associated with it
    * Which is why we have a tuple here, and not two separate values
  - The overhead of distinguishing the different data constructors can be compiled away

- The usual style with newtype is to give the type and constructor the same name

  ```
newtype DifferentTuple = DifferentTuple (Int, String)
  ```

  ```
intFromDiffTup (DifferentTuple (n,_)) = n
  ```

- Optionally, we can also declare an accessor function at the same time

  ```
newtype DifferentTuple
  = DifferentTuple { getDiffTup :: (Int, String) }
  ```

- And type variables are allowed

  ```
newtype ZZ a = ZZ { getZzA :: a }
  ```
Using `newtype` for alternative instances

One use of `newtype` is to associate different instance declarations with a type.

```haskell
newtype WordInt = WI Int

instance Show WordInt where
    show (WI 0) = "zero"
    show (WI 1) = "one"
    show (WI 2) = "two"
    show (WI 3) = "three"
    show (WI 4) = "four"
    show (WI 5) = "five"
    show (WI 6) = "six"
    show (WI 7) = "seven"
    show (WI 8) = "eight"
    show (WI 9) = "nine"
```

Exercise 5.68. Consider the declaration:

```haskell
newtype NewTypeExampleWithInt = { theInt :: Int }
```

What type does Haskell tell you that the function `theInt` has? Create some `NewTypeExampleWithInt` values; how do you use them with `theInt`? What results do you get?

5.4.4 Type class examples

5.4.5 Higher-kinded class type variables

Class `Functor` — things we can map over

- Generalizes the `map` function on lists
- From standard prelude:

  ```haskell
class Functor f where
    fmap :: (a -> b) -> f a -> f b
  ```

- `f` is a type constructor, like `BinaryTree`

  ```haskell
  *Main> :kind Int
  Int :: *
  *Main> :kind BinaryTree
  BinaryTree :: * -> *
  *Main> :kind Either
  Either :: * -> * -> *
  *Main> :kind (->)
  (->) :: * -> * -> *
  ```

- So `Functor` is a class of higher-kinded objects, of type constructors, instead of types
BinaryTree as a Functor

```haskell
ingstance Functor BinaryTree where
  fmap f (Leaf x) = Leaf $ f x
  fmap f (Branch xs1 xs2) = Branch (fmap f xs1) (fmap f xs2)
```

- No constraints on BinaryTree's type arguments
  - There are no explicit type arguments to constrain!

Another type constructor class — Foldable

- We’ve seen this in the type signature for the fold functions
  ```haskell
  *Main> :t foldr
  foldr :: Foldable t => (a -> b -> b) -> b -> t a -> b
  *Main> :t foldl
  foldl :: Foldable t => (b -> a -> b) -> b -> t a -> b
  ```
- The list type constructor `[ ]` is a member of Foldable
- With your functions `binaryTreeFoldl` and `binaryTreeFoldr` we could define:

  ```haskell
  instance Foldable BinaryTree where
    foldl = binaryTreeFoldl
    foldr = binaryTreeFoldr
  ```

6 Lambda calculus

6.1 Syntax

Untyped lambda calculus

- A calculus is a way of writing expressions, plus a way of relating one expression to another
- The lambda calculus lets us discuss many programming language features in a very minimal language
- Terms $M$, $N$, $M'$ etc.:
  - Variables $x$, $x'$, $y$, $z_0$, $z_1$, ...
  - Abstractions $\lambda x. M$
  - Applications $MN$
- Properties of terms
  - Free variables, bound variables
  - Open, closed
  - Values
  - Syntactic identity $\equiv$
- Substitution $M[N/x]$
Exercise 6.1.  Remove all possible parentheses from these expressions so as not to change the meaning of the expressions.

- \(((\lambda x.(\lambda y.((x)y))(\lambda z.(z))))\)
- \((xy)(xz)\)
- \((\lambda x.((\lambda y.(\lambda z.z))y)\)

Exercise 6.2.  Write out the sets of free variables and of bound variables for each of the following expressions.

- \((\lambda x.(\lambda y.zxy))(\lambda z.xzw)\)
- \(\lambda x.xy(xz)\)
- \((\lambda x.(\lambda y.(\lambda z.z))y)\)

Exercise 6.3.  Simplify each of the following expressions, writing them as plain lambda terms without substitutions.

- \(((\lambda x.(\lambda y.zxy))(\lambda z.xzw))\left[\frac{\lambda k.kk}{x}\right]\)
- \((\lambda x.(\lambda y.zxy))(\lambda z.xzw)\left[\frac{\lambda k.kk}{x}\right]\)
- \((\lambda x.xy(xz))\left[\frac{\lambda k.kk}{x}\right]\)

Exercise 6.4.  Apply the hygiene condition to each of the expressions in Exercises 6.2 and 6.3.

Exercise 6.5.  [Barendregt, Sec. 2.2] In combinatory logic one considers a small number of closed terms rather than general lambda expressions. One common system uses three terms, \(I \equiv \lambda x.x\), \(S \equiv \lambda x.\lambda y.\lambda z.xz(\lambda y.z)\), \(K \equiv \lambda x.\lambda y.x\). Show that

- \(I = S K K\)
- \(\lambda x.M = K M\) if \(x \notin \text{fv}(M)\)
- \(\lambda x.M N = S(\lambda x.M)(\lambda y.N)\)

What are the normal forms of these terms:

- \((\lambda y.yyy)((\lambda a.\lambda b.a)I(S S))\)
- \(S S S S S S S S\)

6.2 Reduction

Relating terms by reduction

Three reduction rules

\[(\alpha) \quad \lambda x.M \rightarrow \lambda y.M[y/x] \quad \text{if } x \neq y\]
\[(\beta) \quad (\lambda x.M)N \rightarrow M[N/x]\]
\[(\eta) \quad \lambda x.Mx \rightarrow M \quad \text{if } x \notin \text{fv}(M)\]

- From reduction to equality
- \(\alpha\) equivalence classes, hygienic substitution
- Barendregt’s hygiene condition
- Call-by-value, call-by-name
Exercise 6.6. Identify all of the redexes in the following terms.

- \((\lambda x.\lambda y.yx)(\lambda y. (\lambda z.wz)y)\)
- \(z(\lambda z. (\lambda x.x)(\lambda y.zxy))\)

Exercise 6.7. Apply the hygene condition to each of the expressions in Exercise 6.6.

Exercise 6.8. Reduce each of the expressions in Exercises 6.3 and 6.6:

- To \(\beta\) normal form
- To \(\eta\) normal form
- To \(\beta, \eta\) normal form
- To each of the possible ways of contracting one single redex in each term

Properties of reduction

- Uniqueness of normal forms
- Confluence aka the Church-Rosser property
- Normalization properties
  - Untyped vs. simply-typed calculi
- Subject reduction
- Type soundness

Church encodings

- Booleans
  - \texttt{true} \equiv \lambda m.\lambda n.m
  - \texttt{false} \equiv \lambda m.\lambda n.n
  - \texttt{if} \equiv \lambda p.\lambda m.\lambda n.pmn
- Pairs
  - \texttt{mkpair} \equiv \lambda x.\lambda y.\lambda f.fx y
  - \texttt{fst} \equiv \lambda x.\lambda y.x
  - \texttt{snd} \equiv \lambda x.\lambda y.y
- Numbers
  - \texttt{0} \equiv \lambda f.\lambda x.x
  - \texttt{1} \equiv \lambda f.\lambda x.fx
  - \texttt{2} \equiv \lambda f.\lambda x.f(fx)
  - \texttt{3} \equiv \lambda f.\lambda x.f(f(fx)) and so on
  - \texttt{isZero} \equiv \lambda n.\lambda (\lambda x.\texttt{false})\texttt{true}
  - \texttt{succ} \equiv \lambda n.\lambda f.\lambda x.f(nfx)
- **plus** \( \equiv \lambda n_1.\lambda n_2.\lambda f.\lambda x.n_1 f(n_2 f x) \)
- **times** \( \equiv \lambda n_1.\lambda n_2.\lambda f.\lambda x.n_1 (n_2 f) x \)
- Subtraction, division more complicated but possible
- Then negative numbers, rationals, reals, etc.

**Exercise 6.9.** Define a closed lambda term `and` so that

- `and true true` \( \rightarrow_\beta \) `true`
- `and true false` \( \rightarrow_\beta \) `false`

and so on. Write out the details of each of the four relationships.

Do the same for `or`, `not` and `xor`.

**Exercise 6.10.** Define a partial `signum` operator, which returns `0` for `0`, and `1` for any positive number.

**Exercise 6.11.** Write out the step-by-step details of the following reductions:

- `fst (mkpair 2 3)` \( \rightarrow_\beta \) `2`
- `snd (fst (mkpair (mkpair 3 4) false))` \( \rightarrow_\beta \) `4`
- `succ 4` \( \rightarrow_\beta \) `5`
- `succ (succ 2)` \( \rightarrow_\beta \) `4`
- `plus (plus 2 3) 1` \( \rightarrow_\beta \) `6`
- `iszero 0` \( \rightarrow_\beta \) `true`
- `iszero (succ 1)` \( \rightarrow_\beta \) `false`
- `times 2 (plus 1 2)` \( \rightarrow_\beta \) `6`

**Exercise 6.12.** Define the following functions. Write out reduction sequences to show that each is defined correctly on `1`, `2` and `4`.

- `factorial`
- `sumOneUp x = \sum_{i=1}^{x} i`

**Delta reduction**

- The Church encodings show how to represent various concepts as plain lambda expressions
  - It’s interesting that it’s possible
  - In fact, every computable function can be represented in \( \lambda \)
- But we can also extend the language of terms
  - Extend lambda terms to include various other symbols
  - Specify a function \( \delta \) mapping strings of these symbols to one symbol
  - Then \( \delta \) reduction rewrites applications across a string:
    - If \( \delta(s_1 \cdots s_n) = s_0 \)
    - Then \( (\cdots ((s_1 s_2) s_3) \cdots) s_n \rightarrow_\delta s_0 \)
6.3 Parameter-passing disciplines

Notes to arrive

6.4 Simply-typed λ

Simply-typed lambda calculus

- Environments and judgments
  - An environment is a set of assumptions about how free variables are typed
    * For example: x:Int, f:Int → Int
    * Use upper-case Greek letters for environments: Γ
    * Also called contexts
  - A judgment is a statement that we have proven an expression to be of a particular type
    * Environment "⊢ term : type"

- Simply-typed lambda calculus with integers

\[ \begin{align*}
\text{Var} & \quad \Gamma, x : A \vdash x : A \\
\text{Const} & \quad \Gamma \vdash \mathbf{1} : \text{Int} \quad \text{etc.}
\end{align*} \]

\[ \begin{align*}
\text{Abstr} & \quad \Gamma, x : A \vdash M : B \\
& \quad \Gamma \vdash \lambda x. M : A \rightarrow B
\end{align*} \]

\[ \begin{align*}
\text{Apply} & \quad \Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A \\
& \quad \Gamma \vdash MN : B
\end{align*} \]

- Important properties
  - Progress: If \( \Gamma \vdash M : A \), then either \( M \rightarrow N \), or \( M \) is a value
  - Preservation: If \( \Gamma \vdash M : A \), and \( M \rightarrow N \), then \( \Gamma \vdash N : A \)
  - Normalizing: If \( \Gamma \vdash M : A \), then \( M \) reduces to a normal form in a finite number of steps

Exercise 6.13. For each of the closed terms in Exercises 6.1 through 6.6, determine which are simply-typable (and their types), and which are not. For each of the open terms, which are simply-typable under a suitable environment? In both, write out the typing trees (and environments) for each which is typable; explain why not for the ones which are not.

Pairs and sum types

- Pairs

\[ \begin{align*}
\times & \quad \Gamma \vdash M : A \\
& \quad \Gamma \vdash N : B \\
\end{align*} \]

\[ \begin{align*}
\times E_1 & \quad \Gamma \vdash M : A \times B \\
& \quad \Gamma \vdash \text{fst} M : A
\end{align*} \]

\[ \begin{align*}
\times E_2 & \quad \Gamma \vdash M : A \times B \\
& \quad \Gamma \vdash \text{snd} M : B
\end{align*} \]

- Sums

\[ \begin{align*}
+ & \quad \Gamma \vdash M : A \\
& \quad \Gamma \vdash N : B
\end{align*} \]

\[ \begin{align*}
+E & \quad \Gamma \vdash L : A + B \\
& \quad \Gamma, x : A \vdash M : C \\
& \quad \Gamma, y : B \vdash N : C \\
& \quad \Gamma \vdash \text{case } L \text{ of } x. M | \text{right } y. N : C
\end{align*} \]
Exercise 6.14. What are appropriate reduction rules for expressions with \( \times \) and \(+\) types?

6.5 Polymorphism in lambda calculi

A simple version of polymorphism

- Extend types: could also be a type variable \( \alpha \), or a quantification \( \forall \alpha. T \)

- Rules:
  \[
  \Gamma \vdash M : \forall \alpha. A \quad \text{if } \alpha \notin \text{fv}(\Gamma)
  \]
  \[
  \Gamma \vdash M : A \quad [\alpha \mapsto T] \quad \text{for any type } T
  \]

- More polymorphism than practical
  - Most languages restrict where type variables can be abstracted
  - In Java/Scala, at class/method declarations

Exercise 6.15. Work out closed polymorphic types for each of the Church-encoded terms above. There may be several possible types for each, with quantifiers placed in different positions. Give types with:

- The quantifiers placed as deeply within the type as possible
- All quantifiers to the far outer left of the type

7 Further topics

Time allowing, we will study additional topics at the end of the semester. Any additional notes and exercises will be distributed separately.

8 Hints on selected exercises

Exercise 5.43 (p. 36) Use a helper function which performs the actual recursion. The helper can return both whether the invariants are satisfied, and the sum of the black nodes in the subtree.